

THE CONDENSATION PHASE TRANSITION IN THE REGULAR k -SAT MODEL

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ABSTRACT. Much of the recent work on phase transitions in random discrete structures has been inspired by ingenious but non-rigorous approaches from physics. The physics predictions typically come in the form of distributional fixed point problems that are intended to mimic Belief Propagation, a message passing algorithm. In this paper we propose a novel method for harnessing Belief Propagation directly to obtain a rigorous proof of such a prediction, namely the existence and location of a condensation phase transition in the random regular k -SAT model.

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1. INTRODUCTION

1.1. Background and motivation. Over the past three decades the study of random constraint satisfaction problems has been driven by ideas from statistical physics [22, 23]. This work has had a substantial impact on computer science (e.g., proofs that certain benchmark instances are difficult for certain algorithms), coding theory (“low density parity check codes”) and probabilistic combinatorics (random graphs, hypergraphs and formulas); e.g., [9, 13, 16, 17, 18, 19, 28]. All of these disciplines deal with a common setup. There are a large number of “variables” that interact through a similarly large number of “constraints”. Each variable ranges over a finite domain (such as the Boolean values ‘true’ and ‘false’) and every constraint binds a small number of variables, either encouraging or discouraging certain value combinations.

The striking feature of the physics work is that it is based on a non-rigorous but generic approach called the *cavity method*, centered around the *Belief Propagation* message-passing algorithm, that can be applied almost mechanically [21]. Hence the impact of a single technique on such a wide range of problems. By comparison, the rigorous study of random problems has largely been case-by-case. This begs the question of whether the Belief Propagation calculations can be put on a rigorous basis more directly.

This is precisely the thrust of the present paper. We show how the physics calculations can be turned into a rigorous proof in a highly non-trivial and somewhat representative case. Specifically, we determine the “condensation phase transition” in the random regular k -SAT model. The proof is based on a novel approach that demonstrates how our recent general results on the connection between spatial mixing properties and the computation of the free energy [5] can be put to work. The centrepiece of the proof is a fairly direct analysis of the Gibbs marginals by means of Belief Propagation. The arguments are rather generic and we expect them to extend to other problems.

The random regular k -SAT model is defined as follows [26]. There are Boolean variables x_1, \dots, x_n and m constraints, namely propositional clauses of length k . Each variable occurs precisely $d/2$ times as a positive and precisely $d/2$ times as a negative literal. Hence, $m = dn/(2k)$; we assume tacitly that d is even and that k divides dn . Let $\Phi = \Phi_{d,k}(n)$ signify a uniformly random such k -SAT formula.¹ For k exceeding a certain constant k_0 the threshold where Φ ceases to be satisfiable is known [9].² While the exact formula is cumbersome, asymptotically $d_{k\text{-SAT}}/k = 2^k \ln 2 - k \ln 2/2 + O(1)$ for large k .

Of course, finding the satisfiability threshold is hardly the end of the story. Much more precise information is encoded in the Hamiltonian $\sigma \mapsto E_\Phi(\sigma)$ that maps each truth assignment σ to the number of clauses that it violates. We think of it as a “landscape” on the Hamming cube. For instance, if E_Φ is riddled with local minima, we should expect that Markov processes such as Simulated Annealing get trapped [1, 20, 23]. Hence, E_Φ holds the key to understanding algorithms for finding, counting and sampling solutions [25, 27].

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¹The regular k -SAT model shares many of the properties of the better known model where m clauses are chosen uniformly and independently but avoids the intricacies that result from having a few variables of very high degree.

²In the sense that $\liminf_{n \rightarrow \infty} \mathbb{P}[\Phi \text{ is satisfiable}] > 0$ if $d < d_{k\text{-SAT}}$ and $\lim_{n \rightarrow \infty} \mathbb{P}[\Phi \text{ is satisfiable}] = 0$ if $d > d_{k\text{-SAT}}$.

The key quantity upon which the study of the Hamiltonian hinges is the *partition function*

$$Z_{\Phi} : \beta \in (0, \infty) \mapsto \sum_{\sigma} \exp(-\beta E_{\Phi}(\sigma)).$$

As usual, the larger the *inverse temperature* β , the bigger the relative contribution of “good” assignments that violate few clauses. Of course, we are interested in the asymptotics as $n \rightarrow \infty$. Since $Z_{\Phi}(\beta)$ scales exponentially with n , we consider

$$\phi_{d,k} : \beta \in (0, \infty) \mapsto \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\ln Z_{\Phi}(\beta)]. \quad (1.1)$$

Clearly, what makes $\phi_{d,k}$ vicious is that the log is *inside* the expectation. The existence of the limit follows from the interpolation method [8] and Azuma's inequality implies that $\ln Z_{\Phi}(\beta)$ concentrates about $\mathbb{E}[\ln Z_{\Phi}(\beta)]$.

A key question is how smoothly $\phi_{d,k}(\beta)$ varies as a function of β for *fixed* d, k . Formally, let us call $\beta_0 \in (0, \infty)$ *smooth* if there exists $\varepsilon > 0$ such that the function $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \mapsto \phi_{d,k}(\beta)$ admits an expansion as an absolutely convergent power series around β_0 . If β_0 fails to be smooth, a *phase transition* occurs at β_0 .

1.2. Results. According to (non-rigorous) physics predictions [20] for certain values of d close to the satisfiability threshold $d_{k\text{-SAT}}$ there occurs a so-called *condensation phase transition* at a certain critical $\beta_{\text{cond}}(d, k) > 0$. The main result of this paper proves this conjecture. Let us postpone the precise definition of $\beta_{\text{cond}}(d, k)$ for a moment.

Theorem 1.1. *There exists $k_0 \geq 3$ such that for all $k \geq k_0$, $d \leq d_{k\text{-SAT}}$ there is $\beta_{\text{cond}}(d, k) \in (0, \infty]$ such that any $\beta \in (0, \beta_{\text{cond}}(d, k))$ is smooth. If $\beta_{\text{cond}}(d, k) < \infty$, then there occurs a phase transition at $\beta_{\text{cond}}(d, k)$.*

Thus, if we fix d, k such that $\beta_{\text{cond}}(d, k) = \infty$, then the function $\phi_{d,k}$ is analytic on $(0, \infty)$. But if d, k are such that $\beta_{\text{cond}}(d, k) < \infty$, then $\phi_{d,k}$ is non-analytic at the point $\beta_{\text{cond}}(d, k)$. In fact, we will see that $\beta_{\text{cond}}(d, k) < \infty$ for d exceeding a specific $d_{\text{cond}}(k) < d_{k\text{-SAT}}$. Crucially, Theorem 1.1 identifies the *precise* condensation threshold $\beta_{\text{cond}}(d, k)$; it is the first such result in a model of this kind.

Let us take a look at the precise value of $\beta_{\text{cond}}(d, k)$. As most predictions based on the cavity method, $\beta_{\text{cond}}(d, k)$ results from a *distributional fixed point problem*, i.e., a fixed point problem on the space of probability measures on the unit interval $(0, 1)$. The fixed point problem derives mechanically from the “1RSB cavity equations” [21]. Specifically, writing $\mathcal{P}(\Omega)$ for the set of probability measures on Ω , we define two maps

$$\mathcal{F}_{k,d,\beta} : \mathcal{P}(0, 1) \rightarrow \mathcal{P}(0, 1), \quad \hat{\mathcal{F}}_{k,d,\beta} : \mathcal{P}(0, 1) \rightarrow \mathcal{P}(0, 1)$$

as follows. Given $\pi \in \mathcal{P}(0, 1)$ let $\eta = (\eta_1, \dots, \eta_{k-1}) \in (0, 1)^{k-1}$ be a random $k-1$ -tuple drawn from the distribution $(\hat{z}(\eta) / \hat{Z}(\pi)) d\otimes_{j=1}^{k-1} \pi(\eta_j)$, where

$$\hat{z}(\eta) = 2 - (1 - \exp(-\beta)) \prod_{j < k} \eta_j \quad \text{and} \quad \hat{Z}(\pi) = \int \hat{z}(\eta) d\otimes_{j < k} \pi(\eta_j).$$

Then $\widehat{\mathcal{F}}_{k,d,\beta}(\pi)$ is the distribution of $(1 - (1 - \exp(-\beta)) \prod_{i=1}^{k-1} \eta_i) / \hat{z}(\eta)$. Similarly, given $\hat{\pi} \in \mathcal{P}(0, 1)$ draw $\hat{\eta} = (\hat{\eta}_1, \dots, \hat{\eta}_{d-1})$ from $(z(\hat{\eta}) / Z(\hat{\pi})) d\otimes_{j=1}^{d-1} \hat{\pi}(\hat{\eta}_j)$, where

$$z(\hat{\eta}) = \prod_{j < d/2} \hat{\eta}_j \prod_{j \geq d/2} (1 - \hat{\eta}_j) + \prod_{j < d/2} (1 - \hat{\eta}_j) \prod_{j \geq d/2} \hat{\eta}_j, \quad Z(\hat{\pi}) = \int z(\hat{\eta}) d\otimes_{j < k} \hat{\pi}(\hat{\eta}_j).$$

Then $\mathcal{F}_{k,d,\beta}(\hat{\pi})$ is the distribution of $(\prod_{j < d/2} \hat{\eta}_j \prod_{j \geq d/2} (1 - \hat{\eta}_j)) / z(\hat{\eta})$. Call a distribution $\pi \in \mathcal{P}(0, 1)$ *skewed* if the probability mass of the interval $(0, 1 - \exp(-k\beta/2))$ satisfies $\pi(0, 1 - \exp(-k\beta/2)) < 2^{-0.9k}$.

Proposition 1.2. *Let $d_-(k) = d_{k\text{-SAT}} - k^5$ and $\beta_-(k, d) = k \ln 2 - 10 \ln k$. The map $\mathcal{G}_{k,d,\beta} = \mathcal{F}_{k,d,\beta} \circ \widehat{\mathcal{F}}_{k,d,\beta}$ has a unique skewed fixed point $\pi_{k,d,\beta}^*$, provided that $k \geq k_0$, $d \in [d_-(k), d_{k\text{-SAT}}]$ and $\beta > \beta_-(k, d)$.*

To extract $\beta_{\text{cond}}(d, k)$, let $v_1, \dots, v_k, \hat{v}_1, \dots, \hat{v}_d$ be independent random variables such that the v_i have distribution $\pi_{k,d,\beta}^*$ and the \hat{v}_i have distribution $\widehat{\mathcal{F}}_{k,d,\beta}(\pi_{k,d,\beta}^*)$. Setting

$$z_1 = \prod_{j \leq d/2} \hat{v}_j \prod_{j > d/2} (1 - \hat{v}_j) + \prod_{j \leq d/2} (1 - \hat{v}_j) \prod_{j > d/2} \hat{v}_j, \quad z_2 = 1 - (1 - \exp(-\beta)) \prod_{j \leq k} v_j$$

and $z_3 = v_1 \hat{v}_1 + (1 - v_1)(1 - \hat{v}_1)$, we let

$$\mathcal{F}(k, d, \beta) = \ln \mathbb{E}[z_1] + \frac{d}{k} \ln \mathbb{E}[z_2] - d \ln \mathbb{E}[z_3], \quad \mathcal{B}(k, d, \beta) = \frac{\mathbb{E}[z_1 \ln z_1]}{\mathbb{E}[z_1]} + \frac{d}{k} \frac{\mathbb{E}[z_2 \ln z_2]}{\mathbb{E}[z_2]} - d \frac{\mathbb{E}[z_3 \ln z_3]}{\mathbb{E}[z_3]}. \quad (1.2)$$

Finally, with the usual convention that $\inf \emptyset = \infty$ we let

$$\beta_{\text{cond}}(k, d) = \begin{cases} \infty & \text{if } d < d_-(k), \\ \inf\{\beta > \beta_-(k, d) : \mathcal{F}(k, d, \beta) < \mathcal{B}(k, d, \beta)\} & \text{if } d \in [d_-(k), d_{k-\text{SAT}}]. \end{cases}$$

We proceed to highlight a few consequences of Theorem 1.1 and its proof. The following result shows that $\beta_{\text{cond}}(d, k) < \infty$, i.e., that a condensation phase transition occurs, for degrees d strictly below the satisfiability threshold.

Corollary 1.3. *If $k \geq k_0$, then $d_{\text{cond}}(k) = \min\{d > 0 : \beta_{\text{cond}}(d, k) < \infty\} < d_{k-\text{SAT}} - \Omega(k)$.*

Furthermore, the following corollary shows that the so-called “replica symmetric solution” predicted by the cavity method yields the correct value of $\phi_{d,k}(\beta)$ for $\beta < \beta_{\text{cond}}(d, k)$.

Corollary 1.4. *If $k \geq k_0$, $d \leq d_{k-\text{SAT}}$ and $\beta < \beta_{\text{cond}}(d, k)$, then $\phi_{d,k}(\beta) = \mathcal{F}(k, d, \beta)$.*

Corollary 1.4 opens the door to studying the “landscape” E_{Φ} for $\beta < \beta_{\text{cond}}(d, k)$. Specifically, Corollary 1.4 enables us to bring the “planting trick” from [1] to bear so that we can analyse typical properties of samples from the Gibbs measure. We leave a detailed discussion to future work. Finally, complementing Corollary 1.4, the following result shows that $\mathcal{F}(k, d, \beta)$ overshoots $\phi_{d,k}(\beta)$ for $\beta > \beta_{\text{cond}}(d, k)$.

Corollary 1.5. *If $k \geq k_0$, $d \leq d_{k-\text{SAT}}$ and $\beta > \beta_{\text{cond}}(d, k)$, then there is $\beta_{\text{cond}}(d, k) < \beta' < \beta$ such that $\phi_{d,k}(\beta') < \mathcal{F}(k, d, \beta')$.*

1.3. Outline and related work. Admittedly, the definition of $\beta_{\text{cond}}(k, d)$ is not exactly simple. For instance, even though the fixed point distribution from Proposition 1.2 stems from a discrete problem, it turns out to be a continuous distribution on $(0, 1)$. Yet perhaps despite appearances, the analytic formula (1.2) is conceptually *far* simpler than the definition of $\phi_{d,k}$. For instance, we are going to see in Section 2 that the fixed point problem can be understood elegantly in terms of a Galton-Watson tree. Thus, one could say that Theorem 1.1 reduces the condensation problem on the complex random formula Φ to a problem on a random tree.

The proof of Theorem 1.1 builds upon an abstract result from [5] that, roughly speaking, reduces the study of the partition function to two tasks. First, to calculate the marginals of the Gibbs measure induced by a random formula $\hat{\Phi}$ chosen from a reweighted probability distribution, the “planted model”. Second, to prove that the Gibbs measure of $\hat{\Phi}$ enjoys the non-reconstruction property, a spatial mixing property. The technical contribution of the present work is to actually tackle these two tasks problems in a fairly generic way. Our principal tool is going to be the Belief Propagation algorithm, the cornerstone of the physicists’ cavity method. In particular, we are going to see that the distributional operator $\mathcal{G}_{k,d,\beta}$ from Proposition 1.2 mimics Belief Propagation run on a Galton-Watson tree that captures the local geometry of the formula $\hat{\Phi}$. The predictions of the “cavity method” typically come as distributional fixed points but there are few proofs that establish such predictions rigorously. The one most closely related to the present work is the paper of Bapst et al. [6] on condensation in random graph coloring. It determines the critical average degree d for which condensation starts to occur with respect to the number of proper k -colorings of the Erdos-Rényi random graph. Conceptually, this corresponds to taking the limit $\beta \rightarrow \infty$ in (1.1), which simplifies the problem rather substantially. Thus, the main result of [6] corresponds to Corollary 1.3. Other previous results on condensation, which dealt with random hypergraph 2-coloring and the Potts model on the random graph, were only approximate [7, 11, 12].

Interestingly, determining the satisfiability threshold on the random regular formula Φ is conceptually much easier than identifying the condensation threshold [9]. This is because the local structure of the random formula Φ is essentially deterministic, namely a tree comprising of clauses and variables in which every variable appears $d/2$ times positively and $d/2$ times negatively. In effect, the satisfiability threshold is given by a fixed point problem on the unit interval rather than on the space of probability measures on the unit interval. Similar simplifications occur in other regular models [14, 15], and these proofs employed Belief Propagation in this simpler setting. By contrast, we will see in Section 2 that the condensation phase transition hinges on the reweighted distribution $\hat{\Phi}$, whose local structure is genuinely random.

Recent work on the k -SAT threshold in uniformly random formulas [9, 10], in particular the breakthrough paper by Ding, Sly and Sun [16], also harnessed the physicists’ Belief Propagation or Survey Propagation calculations.³ In the uniformly random model a substantial technical difficulty is posed by the presence of variables of exceptionally

³Survey Propagation can be viewed as a Belief Propagation applied to a modified constraint satisfaction problem [21].

high degree, an issue that is, of course, absent in the regular model. Specifically, [9, 10, 16] apply the second moment method to a random variable whose construction is guided by Belief/Survey Propagation. By contrast, here we employ Belief Propagation in the more direct way enabled by [5].

1.4. Notation and preliminaries. We generally view a regular k -SAT instance Φ as bijections from sets of clause clones to sets of variable clones (“configuration model”). That is, given n, m, d, k , we let $\{x_1, \dots, x_n\} \times [d]$ be the set of variable clones and $\{a_1, \dots, a_m\} \times [k]$ the set of clause clones. Then $\Phi : \{x_1, \dots, x_n\} \times [d] \rightarrow \{a_1, \dots, a_m\} \times [k]$ is a bijection. The first $d/2$ clones of each variable are considered its positive occurrences and the last $d/2$ ones its negative occurrences.

We denote the image of a clone (x_i, j) by $\partial_\Phi(x_i, j)$ and the inverse image of (a_i, j) by $\partial_\Phi(a_i, j)$. Analogously, $\partial_\Phi^\ell(v, j)$ is the depth- ℓ neighborhood of clone (v, j) . Moreover, we define Φ as a uniformly random bijection. By standard arguments this distribution is easily seen to be contiguous to the uniform distribution on regular formulas.

Suppose that the variables and clauses of Φ, Φ' are x_i, x'_i, a_j, a'_j for $i \in [n], j \in [m]$. We distinguish (variable or clause) clones r, r' of Φ, Φ' , which we consider their roots. An *isomorphism* $\psi : \Phi \rightarrow \Phi'$ is a bijection with the following properties.

ISM1: $r' = \psi(r)$.

ISM2: ψ maps variable clones to variable clones and clause clones to clause clones.

ISM3: If $\psi(v, h) = (w, j)$, then $h = j$.

ISM4: We have $\psi \circ \Phi(v, h) = \Phi' \circ \psi(v, h)$ for all clones (v, h) .

Let $\ell \geq 0$ and let T be a regular k -SAT formula with a distinguished (variable or clause) clone r . For each variable clone (x, i) of Φ we have a random variable $\mathbf{1}\{\partial^\ell T \cong \partial_\Phi^\ell(x, i)\}$ that indicates that the depth- ℓ neighborhood of Φ rooted at (x, i) is isomorphic to T . Similarly, for each clause cone (a, j) of Φ we consider the random variable $\mathbf{1}\{\partial^{\ell+1} T \cong \partial_\Phi^{\ell+1}(a, j)\}$. Let \mathfrak{T}_ℓ be the σ -algebra generated by all these random variables. Thus, \mathfrak{T}_ℓ captures the “local structure” of the random formula up to depth ℓ .

2. OUTLINE

2.1. Two moments do not suffice. The default approach to studying the function $\phi_{d,k}(\beta)$ is the venerable “second moment method”. Cast on a logarithmic scale, if

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}[Z_\Phi(\beta)^2] \leq \lim_{n \rightarrow \infty} \frac{2}{n} \ln \mathbb{E}[Z_\Phi(\beta)], \quad \text{then} \quad (2.1)$$

$$\phi_{d,k}(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}[Z_\Phi(\beta)]. \quad (2.2)$$

The last term is easy to study because the log is outside the expectation. In particular, the function $\beta \in (0, \infty) \mapsto \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}[Z_\Phi(\beta)]$ turns out to be analytic. Consequently, the least $\beta \in (0, \infty)$ where (2.2) fails to hold must be a phase transition.

From a bird’s eye view, both the physics intuition and the second moment are all about the geometry of the *Gibbs measure* of Φ at a given $\beta \in (0, \infty)$. Let us encode truth assignments as points $\sigma \in \{\pm 1\}^n$ with the convention that 1 stands for ‘true’ and -1 for ‘false’. Then the Gibbs measure is the distribution on $\{\pm 1\}^n$ defined by

$$\sigma \in \{\pm 1\}^n \mapsto \exp(-\beta E_\Phi(\sigma)) / Z_\Phi(\beta).$$

Thus, we weigh assignments according to the number of clauses that they violate, giving greater weight to ‘better’ assignments as β gets larger. Let $\sigma, \sigma_1, \sigma_2, \dots$ be independent samples from the Gibbs measure and write $\langle X(\sigma_1, \dots, \sigma_l) \rangle_{\Phi, \beta}$ for the expectation of $X : (\{\pm 1\}^n)^l \rightarrow \mathbb{R}$. Then according to the physics picture the condensation point $\beta_{\text{cond}}(k)$ should be the supremum of all $\beta > 0$ such that $\mathbb{E} \langle |\sigma_1 \cdot \sigma_2| \rangle_{\Phi, \beta} = o(n)$. In other words, if we choose a random formula Φ and then sample two assignments σ_1, σ_2 according to the Gibbs measure independently, then σ_1, σ_2 will be about orthogonal. This decorrelation property is, roughly speaking, a *necessary* condition for the success of the second moment method as well [2, 4]. Therefore, the prediction that $\mathbb{E} \langle |\sigma_1 \cdot \sigma_2| \rangle_{\Phi, \beta} = o(n)$ right up to $\beta_{\text{cond}}(d, k)$ may inspire confidence that the same is true of (2.1). In fact, we will prove in Section 7 that (2.1) holds if either d or β is relatively small.

Lemma 2.1. *If $d \leq d_-(k)$ or $\beta \leq \beta_-(k, d)$ then (2.1) is true.*

However, for β near $\beta_{\text{cond}}(d, k)$ the second moment method turns out to fail rather spectacularly. Formally, if $\beta_{\text{cond}}(d, k) < \infty$, then there exists $\varepsilon > 0$ such that (2.1) is violated for all $\beta > \beta_{\text{cond}}(d, k) - \varepsilon$, i.e., the second moment overshoots the square of the first moment by a factor that is exponential in n .

2.2. Quenching the average. To understand what goes awry it is convenient to turn the second moment into a first moment under a reweighted distribution that we call the *planted model*. This is the distribution on formula/assignment pairs under which the probability of $(\hat{\Phi}, \hat{\sigma})$ equals $\exp(-\beta E_{\hat{\Phi}}(\hat{\sigma})) / \mathbb{E}[Z_{\Phi}(\beta)]$. Let $(\hat{\Phi}, \hat{\sigma})$ be a random pair drawn from this distribution. Then by symmetry the distribution of the assignment $\hat{\sigma}$ is uniform and we may assume without loss that $\hat{\sigma} = \mathbf{1}$ is the all-ones assignment. Further, the probability that a specific formula $\hat{\Phi}$ comes up equals $\mathbb{P}[\hat{\Phi} = \hat{\Phi}] = Z_{\beta}(\hat{\Phi}) / \mathbb{E}[Z_{\beta}(\Phi)]$. Thus, the planted distribution weighs formulas by their partition function. In effect,

$$\mathbb{E}[Z_{\Phi}(\beta)^2] = \mathbb{E}[Z_{\Phi}(\beta)] \cdot \mathbb{E}[Z_{\hat{\Phi}}(\beta)].$$

If we go over the proof of Lemma 2.1, we see that $\mathbb{E}[Z_{\hat{\Phi}}(\beta)]$ is dominated by two distinct contributions. First, assignments that are more or less orthogonal to $\hat{\sigma}$ yield a term of order $\mathbb{E}[Z_{\Phi}(\beta)]$. Second, there is a contribution from σ close to $\hat{\sigma} = \mathbf{1}$; say, $\sigma \cdot \mathbf{1} \geq n(1 - 2^{-k/10})$. Geometrically, this reflects the fact that the planted assignment $\mathbf{1}$ sits in a “valley” of the Hamiltonian $E_{\hat{\Phi}}$ w.h.p. The valleys are known as clusters in the physics literature and we let

$$\mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta) = Z_{\hat{\Phi}}(\beta) \langle \mathbf{1} \{ \sigma \cdot \mathbf{1} > n(1 - 2^{-k/10}) \} \rangle_{\hat{\Phi}, \beta}$$

be the (weighted) *cluster size*. Performing an elementary calculation, we find that it is the expected cluster size that derails the second moment method for β near $\beta_{\text{cond}}(d, k)$.

At a second glance, this is unsurprising. For $\mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)$ scales exponentially with n and is therefore prone to large deviations effects. To suppress these we ought to investigate $\mathbb{E}[\ln \mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)]$ instead of $\mathbb{E}[\mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)]$. A similar issue (that the expected cluster size drives up the second moment) occurred in earlier work on condensation [6, 7, 11, 12]. Borrowing the remedy suggested in these papers, we prove in Section 7 that applying the second moment method to a carefully truncated random variable yields

Lemma 2.2. *Equation (2.2) holds iff*

$$\limsup_{n \rightarrow \infty} n^{-1} \mathbb{E}[\ln \mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)] \leq \lim_{n \rightarrow \infty} n^{-1} \ln \mathbb{E}[Z_{\hat{\Phi}}(\beta)]. \quad (2.3)$$

Computing $\lim_{n \rightarrow \infty} n^{-1} \ln \mathbb{E}[Z_{\hat{\Phi}}(\beta)]$ is easy, as the following standard lemma shows.

Lemma 2.3. *Assume that $d \leq d_{k-\text{SAT}}$ and $\beta \in \mathbb{R}$. Then $\lim_{n \rightarrow \infty} n^{-1} \ln \mathbb{E}[Z_{\hat{\Phi}}(\beta)] = \mathcal{F}(k, d, \beta)$.*

Hence, we are left to calculate $\mathbb{E}[\ln \mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)]$, the “quenched average” in physics jargon. As the log and the expectation do not commute, this problem is well beyond the reach of elementary methods. Tackling it is the main achievement of this paper. Specifically, we are going to prove

Proposition 2.4. *Assume that $d \in [d_-(k), d_{k-\text{SAT}}]$ and $\beta > \beta_-(k, d)$. Then $\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[\ln \mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)] = \mathcal{B}(k, d, \beta)$.*

2.3. Non-reconstruction and the Bethe free energy. In the following we let for a formula Φ and $v \in V \cup F$ and $\ell \geq 0$, $\partial_{\Phi}^{\ell} v$ (resp. $\Delta_{\Phi}^{\ell} v$) denote the set of vertices at distance exactly ℓ (resp. less than ℓ) from v in Φ .

To prove Proposition 2.4 we investigate the spatial mixing properties of the conditional Gibbs measure

$$\mathbb{I}_{\hat{\Phi}, \beta} = \left\langle \cdot \middle| \mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta) \right\rangle_{\hat{\Phi}, \beta}.$$

Specifically, for a variable x , an assignment $\sigma \in \mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)$ and an integer $\ell \geq 0$ let $\nabla(\hat{\Phi}, x, \ell)$ be the σ -algebra on $\mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)$ generated by the random variables $\sigma(y)$ for variable y at distance greater either ℓ or $\ell + 1$ from x . Further, we define

$$\mu_{\hat{\Phi}, x}^{(\ell)}(\pm 1) = \mathbb{I}_{\hat{\Phi}, \beta}[\mathbf{1}\{\sigma(x) = \pm 1\} | \nabla(\hat{\Phi}, x, \ell)]_{\hat{\Phi}, \hat{\sigma}}(\hat{\sigma}). \quad (2.4)$$

In words, $\mu_{\hat{\Phi}, x}^{(\ell)}(\pm 1)$ is the probability that x gets assigned ± 1 in a random assignment of its depth- ℓ neighborhood under the boundary condition induced by $\hat{\sigma}$.

We lift the distributions from (2.4) to clauses. In slightly greater generality, suppose that μ is a map that assigns each variable x a probability distribution $\mu_x \in \mathcal{P}(\{\pm 1\})$. Then for clause a of a formula $\hat{\Phi}$ we let $\mu_{\hat{\Phi}, a}^{(2\ell+1)}$ be the distribution on $\{\pm 1\}^k$ with the following two properties.

- (i) if $j \in [k]$ and $x = \partial_{\hat{\Phi}}(a, j)$, then the marginal distribution of the j th coordinate coincides with μ_x .

(ii) subject to (i), $H(\mu_{\hat{\Phi},a}^{(2\ell+1)}) + \langle \ln \psi_a \rangle_{\mu_{\hat{\Phi},a}^{(2\ell+1)}}$ is maximum.

These two conditions determine $\mu_{\hat{\Phi},a}^{(2\ell+1)}$ uniquely (because the entropy is concave).

Now, we say that a formula $\hat{\Phi}$ has the *non-reconstruction property* if for any $\varepsilon > 0$ there is $\ell > 0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \sum_x \left[\left| \mu_{\hat{\Phi},x}^{(2\ell)}(1) - \|\sigma(x) | \nabla(\hat{\Phi}, x, 2\ell)\|_{\hat{\Phi},\beta} \right| \right]_{\hat{\Phi},\beta} < \varepsilon \right] = 1$$

The first half of the proof of Proposition 2.4 consist in proving the following.

Proposition 2.5. *Assume that $d \in [d_-(k), d_{k-SAT}]$ and $\beta > \beta_-(k, d)$. Then $\hat{\Phi}$ has the non-reconstruction property.*

Together with results from [5] Proposition 2.5 implies an upper bound on $\mathbb{E}[\ln \mathcal{C}_{\hat{\Phi},\hat{\sigma}}(\beta)]$. But to obtain a matching lower bound a little more work is needed. Specifically, we need to consider a further distribution on formula/assignment pairs that we call the *planted replica model* $(\tilde{\Phi}, \tilde{\sigma})$ generated by the following experiment.

PR1: Choose a random formula $\hat{\Phi}$.

PR2: For each variable x choose $\tilde{\sigma}(x)$ from $\mu_{\hat{\Phi},x}^{(2\ell)}$ independently.

PR3: For every clause a choose $\tilde{\sigma}(a) \in \{\pm 1\}^k$ independently from $\mu_{\hat{\Phi},a}^{(2\ell+1)}$.

PR4: Choose $\tilde{\Phi}$ uniformly at random subject to the following conditions.

- If (a, j) is a clause clone and $\partial_{\hat{\Phi}}(a, j) = (x, i)$, then $\tilde{\sigma}(a, j) = \tilde{\sigma}(x)$.
- For all clause clones (a, j) we have $\Delta_{\tilde{\Phi}}^{4\ell+1}(a, j) \cong \Delta_{\hat{\Phi}}^{4\ell+1}(a, j)$.

If no such $\tilde{\Phi}$ exists, start over from **PR2**.

The *planted replica model* has the non-reconstruction property if for any $\varepsilon > 0$ there is $\ell > 0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \sum_x \left[\left| \mu_{\tilde{\Phi},x}^{(2\ell)}(1) - \|\sigma(x) | \nabla(\tilde{\Phi}, x, 2\ell)\|_{\tilde{\Phi},\beta} \right| \right]_{\tilde{\Phi},\beta} < \varepsilon \right] = 1$$

Proposition 2.6. *Assume that $d \in [d_-(k), d_{k-SAT}]$ and $\beta > \beta_-(k, d)$. Then the planted replica model has the non-reconstruction property.*

The non-reconstruction property enables us to determine $\mathbb{E}[\ln \mathcal{C}_{\hat{\Phi},\hat{\sigma}}(\beta)]$. Indeed, given a map $\mu : x \mapsto \mu_x \in \mathcal{P}(\{\pm 1\})$ that assigns each variable a distribution on ± 1 we define the *Bethe free energy* of a formula $\hat{\Phi}$ as

$$\mathcal{B}_{\hat{\Phi}}(\mu) = \sum_x (1-d)H(\mu_x) + \sum_a \left[H(a) + \langle \ln \psi_a \rangle_{\mu_a} \right].$$

Of course, μ_a is the extension of μ from variables to clauses as defined above. We also let $\mathcal{B}_{\hat{\Phi},\ell} = \mathcal{B}_{\hat{\Phi}}((\mu_{\hat{\Phi},x}^{(2\ell)})_{x \in V})$. Then by combining Propositions 2.5 and 2.6 with [5, Theorems 4.4 and 4.5] we obtain the following.

Corollary 2.7. *We have $\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\ln \mathcal{C}_{\hat{\Phi},\hat{\sigma}}(\beta)] = \lim_{\ell \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\mathcal{B}_{\hat{\Phi},\ell}]$.*

Furthermore, $\mathcal{B}_{\hat{\Phi},\ell}$ is determined by the *local* structure of $\hat{\Phi}$. Since $\hat{\Phi}$, the local structure of the random formula can be described in terms of a random tree. In fact, tracing the Belief Propagation algorithm on this random tree enables us to relate $\mathbb{E}[\mathcal{B}_{\hat{\Phi},\ell}]$ to the distributional fixed point problem from Proposition 1.2. The result of this, derived in Section 4, is

Proposition 2.8. *We have $\lim_{\ell \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\mathcal{B}_{\hat{\Phi},\ell}] = \mathcal{B}(k, d, \beta)$.*

Finally, Proposition 2.4 follows from Corollary 2.7 and Proposition 2.8.

3. BELIEF PROPAGATION ON RANDOM TREES

In the following of the paper we assume that $d \in [d_-(k), d_{k-SAT}]$ and that $\beta \geq \beta_-(k, d)$. We let $c_\beta = 1 - \exp(-\beta) \in (0, 1)$.

In this section, we introduce a Galton-Watson process on trees, that will describe the local neighborhood of randomly chosen vertices in random formulas but also allow to analyze the probabilistic fixed point problem in Section 4.

3.1. A Galton-Watson process on trees. We consider the following Galton-Watson process. We first observe that there is a unique $q = q(k, d, \beta) \in (0, 1)$ such that

$$1 - (1 - \exp(-\beta))q^k = 2(1 - q). \quad (3.1)$$

We start from the tree $T_{2\ell}$ of depth 2ℓ ($\ell \geq 0$) such that

- Each node at an even depth of the tree is a variable node and has for offspring $d - 1$ clause nodes.
- Each node at an odd depth of the tree is a clause node and has for offspring $k - 1$ variable nodes.

Let $V_{2\ell}$ be the set of $T_{2\ell}$ variable nodes, $F_{2\ell}$ be the set of its clause nodes. Let $\partial V_{2\ell}$ denote the subset of variables at distance 2ℓ from the root, and for node v of $T_{2\ell}$, let ∂v (resp. $\partial_{\downarrow} v$) denote the set of neighbors (resp. children) of v in $T_{2\ell}$. We further decorate $T_{2\ell}$ as follows. Each node $v \in V_{2\ell} \cup F_{2\ell}$ carries a number $b_{v,\uparrow} \in \{-1, 1\}$ determined by the following process.

- For the root r , we have $b_{r,\uparrow} = 1$ with probability q and $b_{r,\uparrow} = -1$ with probability $1 - q$.
- The offspring of a variable node x with $b_{x,\uparrow} = \pm 1$ is $\frac{d}{2} - 1$ clause nodes a with $b_{a,\uparrow} = \pm 1$ and $\frac{d}{2}$ clause nodes a such that $b_{a,\uparrow} \mp 1$.
- If a clause node a is such that $b_{a,\uparrow} = -1$, the number of $x \in \partial_{\downarrow} a$ such that $b_{x,\uparrow} = -1$ has distribution $\text{Bin}(k - 1, 1 - q)$.
- If a clause node a is such that $b_{a,\uparrow} = 1$, then with probability $\exp(-\beta)q^{k-1}/(1 - (1 - \exp(-\beta))q^{k-1})$ the offspring is $k - 1$ variables x with $b_{x,\uparrow} = 1$, and otherwise the number of $x \in \partial_{\downarrow} a$ such that $b_{x,\uparrow} = -1$ has a conditionnal distribution $\text{Bin}_{\geq 1}(k - 1, 1 - q)$.

Then we define for a clause a and $x \in \partial a$, $b_{a,x} = b_{x,\uparrow}$ if $x \in \partial_{\downarrow} a$ and $b_{a,x} = b_{a,\uparrow}$ otherwise. We let $\partial_{\pm 1} a = \{x \in \partial a, b_{a,x} = \mp 1\}$ and for a variable x , $\partial_{\pm 1} x = \{a \in \partial x, b_{a,x} = \mp 1\}$. We finally let, for $0 \leq l \leq k$, $\partial_{\pm 1, l} x = \{a \in \partial_{\pm 1} x, |\{y \in \partial a \setminus \{x\}, b_{a,y} = 1\}| = l\}$.

Let $T = T(d, k, \beta, 2\ell)$ be the resulting random (decorated) tree, let $p_{k,d,\beta}^{(2\ell)}$ denote its distribution, and let $\mathcal{T}_{2\ell}$ denote the support of $p_{k,d,\beta}^{(2\ell)}$. Similarly, we denote by \hat{T} the random tree pending below the first clause adjacent to the root of $T(d, k, \beta, 2\ell + 2)$, by $\hat{V}_{2\ell+1}$ its set of variables and by $\partial \hat{V}_{2\ell+1}$ its set of variable at distance $2\ell + 1$ from the root, by $\hat{p}_{k,d,\beta}^{(2\ell+1)}$ its distribution, and by $\hat{\mathcal{T}}_{2\ell+1}$ the support of $\hat{p}_{k,d,\beta}^{(2\ell+1)}$.

We call a sequence $\partial v \in \mathcal{P}(\{-1, 1\})^{\partial V_{2\ell}}$ a *boundary condition* over $T \in \mathcal{T}_{2\ell}$. Similarly, for $\hat{T} \in \hat{\mathcal{T}}_{2\ell+1}$, we call a sequence $\partial \hat{v} \in \mathcal{P}(\{-1, 1\})^{\partial \hat{V}_{2\ell+1}}$ a boundary condition on \hat{T} .

We define the Belief Propagation messages induced by the boundary condition ∂v on T as the families $(v_{x,\uparrow}^{T,\partial v})_{x \in V_{2\ell}}$ and $(\hat{v}_{a,\uparrow}^{T,\partial v})_{a \in F_{2\ell}}$, where $v_{x,\uparrow}^{T,\partial v} = (\partial v)_x$ for $x \in \partial V_{2\ell}$, and otherwise

$$v_{x,\uparrow}^{T,\partial v}(s) = \frac{\prod_{a \in \partial_{\downarrow} x} \hat{v}_{a,\uparrow}^{T,\partial v}(s)}{\sum_{s' \in \{-1, 1\}} \prod_{a \in \partial_{\downarrow} x} \hat{v}_{a,\uparrow}^{T,\partial v}(s')} \quad \text{for } x \in V_{2\ell} \setminus \partial V_{2\ell} \text{ and } s \in \{-1, 1\}, \quad (3.2)$$

$$\hat{v}_{a,\uparrow}^{T,\partial v}(s) = \frac{\sum_{s_a \in \{-1, 1\}^{\partial a}} \mathbf{1}_{s_x = s} \psi_{a,\beta}(s_a) \prod_{y \in \partial_{\downarrow} a} v_{y,\uparrow}^{T,\partial v}(s_y)}{\sum_{s_a \in \{-1, 1\}^{\partial a}} \psi_{a,\beta}(s_a) \prod_{y \in \partial_{\downarrow} a} v_{y,\uparrow}^{T,\partial v}(s_y)} \quad \text{for } a \in F_{2\ell} \text{ and } s \in \{-1, 1\}. \quad (3.3)$$

In the following of this section, we let ℓ large enough be fixed. We will be interested in showing that, under reasonable assumptions, the message exiting a tree $T \in \mathcal{T}_{2\ell}$ only weakly depends on the boundary condition ∂v . More precisely, we define $\partial v^{(0)} \in \mathcal{P}(\{-1, 1\})^{\partial V_{2\ell}}$ by $\partial v_x^{(0)}(1) = 1 = 1 - \partial v_x^{(0)}(-1)$ for all $x \in \partial V_{2\ell}$. For $\hat{T} \in \hat{\mathcal{T}}_{2\ell+1}$ we define $\partial \hat{v}^{(0)} \in \mathcal{P}(\{-1, 1\})^{\partial \hat{V}_{2\ell+1}}$ similarly. For a tree $T \in \mathcal{T}_{2\ell}$ with root r , and a boundary condition ∂v on T , we denote by

$$v_T^{\partial v} = v_{r,\uparrow}^{T,\partial v}, \quad v_T^{(2\ell)} = v_{r,\uparrow}^{T,\partial v^{(0)}}.$$

Similarly for $\hat{T} \in \hat{\mathcal{T}}_{2\ell+1}$ with root r and a boundary condition $\partial \hat{v}$ on \hat{T} we let

$$\hat{v}_T^{\partial \hat{v}} = \hat{v}_{r,\uparrow}^{T,\partial \hat{v}}, \quad \hat{v}_T^{(2\ell+1)} = \hat{v}_{r,\uparrow}^{T,\partial \hat{v}^{(0)}}.$$

We are now ready to state the main results of this section. In the following, we denote by T a random tree drawn from the distribution $p_{k,d,\beta}^{(2\ell)}$, and by ∂v a random boundary condition, independent of T and that satisfies the following condition.

$$\mathbf{H} \text{ For any } x \in \partial V_{2\ell}, \mathbb{P} \left[(\partial v)_x(1) \leq 1 - \exp(-k\beta/2) \mid ((\partial v)_y)_{y \in \partial V_{2\ell} \setminus \{x\}} \right] \leq 2^{-0.9k}$$

Similarly, we denote by \hat{T} a random tree drawn from the distribution $\hat{p}_{k,d,\beta}^{(2\ell+1)}$, and by $\partial\mathbf{v}$ a random boundary condition, independent of \hat{T} and that satisfies the following condition.

$$\mathbf{H} \text{ For any } x \in \partial V_{2\ell+1}, \mathbb{P} \left[(\partial\mathbf{v})_x(1) \leq 1 - \exp(-k\beta/2) \mid ((\partial\mathbf{v})_y)_{y \in \partial V_{2\ell+1} \setminus \{x\}} \right] \leq 2^{-0.9k}$$

Proposition 3.1. *We have*

$$\begin{aligned} \mathbb{P} \left[\|v_T^{\partial\mathbf{v}} - v_T^{(2\ell)}\|_\infty \geq 2\ell^{-1} \right] &\leq \ell^{-1}, \\ \mathbb{P} \left[\|\hat{v}_{\hat{T}}^{\partial\mathbf{v}} - \hat{v}_{\hat{T}}^{(2\ell+1)}\|_\infty \geq 2k^2 \exp(2\beta)\ell^{-1} \right] &\leq k^2 \ell^{-1}. \end{aligned}$$

The following variant of the proposition will follow from similar steps, and will prove usefull when analyzing random graphs in Sec 5-6. We first need to slightly generalize the process considered up to now. Let $\text{GW}(k, d, \beta, 2\ell)$ denote the Galton-Watson process introduced considered up to now. Let $\text{GW}'(k, d, \beta, 2\ell)$ be the multi-type random process defined by the same rules as $\text{GW}(k, d, \beta, 2\ell)$, except for the first and second rule which are replaced by

- (i)' The root r has for offspring $\frac{d}{2}$ clauses nodes a with $b_{a,\uparrow} = 1$ and $\frac{d}{2}$ clauses nodes a with $b_{a,\downarrow} = -1$.
- (ii)' The offspring of a variable node x different from the root with $b_{x,\uparrow} = \pm 1$ is $\frac{d}{2} - 1$ clause nodes a with $b_{a,\uparrow} = \pm 1$ and $\frac{d}{2}$ clause nodes a such that $b_{a,\uparrow} \mp 1$.

Let $\tilde{\mathcal{T}}_{2\ell}$ denote the set of trees generated by the process and by $\tilde{p}_{k,d,\beta}^{(2\ell)}$ the associated probability distribution. Let, in the following, T' denote a random tree drawn from this distribution. Let $\partial V'_{2\ell}$ denote the variables at distance 2ℓ from T' root (which is, as previously, a deterministic quantity). Let us call as before $\partial\mathbf{v} \in \mathcal{P}(\{-1, 1\})^{\partial V'_{2\ell}}$ a boundary condition over T' , and let us extend condition **H** into

$$\mathbf{H}' \text{ For any } x \in \partial V'_{2\ell}, \mathbb{P} \left[(\partial\mathbf{v})_x(1) \leq 1 - \exp(-k\beta/2) \mid ((\partial\mathbf{v})_y)_{y \in \partial V'_{2\ell} \setminus \{x\}} \right] \leq 2^{-0.9k}$$

Proposition 3.2. *Assume that $\partial\mathbf{v}''$ is a random boundary condition over $\partial V'_{2\ell}$, independent of T' and that satisfies **H'**. the following assumption. Further assume that the random boundary condition $\partial\mathbf{v}'$ (whose distribution may also depend on T') satisfies for all $x \in \partial V'_{2\ell}$, $\partial\mathbf{v}'_x(1) \geq \partial\mathbf{v}''_x(1)$. Then*

$$\mathbb{P} \left[\|\mu_{T'}^{\partial\mathbf{v}'} - \mu_{T'}^{(2\ell)}\|_\infty \geq k \exp(\beta)\ell^{-1} \right] \leq \ell^{-1}.$$

3.2. The (random) trunk of random trees : proof of Proposition 3.1. In order to prove Proposition 3.1, we shall identify a concrete condition on $(T, \partial\mathbf{v})$ under which the message $v_T^{\partial\mathbf{v}}$ is close to $v_T^{(2\ell)}$. We define the *Trunk* of T under the boundary condition $\partial\mathbf{v}$, $\text{Trunk}(T, \partial\mathbf{v})$, as the largest subset W of $V_{2\ell}$ such that for any $x \in W$ either

TR0: $x \in \partial V_{2\ell}$ and $\partial\mathbf{v}_x(1) \geq 1 - \exp(-k\beta/2)$

or the five following conditions hold

TR1: there are at least $\lfloor 0.9k \rfloor$ clauses $a \in \partial_1 x$ such that $\partial_1 a = \{x\}$.

TR2: there are no more than $\lfloor 0.1k \rfloor$ clauses $a \in \partial x$ such that $|\partial_{-1} a| = k$.

TR3: for any $1 \leq l \leq k$ the number of $a \in \partial_{-1} x$ such that $|\partial_1 a| = l$ is bounded by $k^{l+3}/l!$.

TR4: there are no more than $k^{3/4}$ clauses $a \in \partial_1 x$ such that $|\partial_1 a| = 1$ but $\partial a \not\subseteq W$.

TR5: there are no more than $k^{3/4}$ clauses $a \in \partial_{-1} x$ such that $|\partial_{-1} a| < k$ and $|\partial_1 a \setminus W| \geq |\partial_1 a|/4$.

We will first observe that the following is true.

Lemma 3.3. *We have*

$$\mathbb{P} \left[\text{the root of } T \text{ under the boundary condition } \partial\mathbf{v} \text{ is cold} \right] \geq 1 - \ell^{-1}.$$

Proof. The lemma is easily proved by induction. □

Further, we need to introduce the following definitions. For $T \in \tilde{\mathcal{T}}_{2\ell}$ with root r and $x \in \partial V_{2\ell}$, we denote by $[x \rightarrow r]$ the unique shortest path from x to r in T .

- (i) We say that a factor node $a \in F_{2\ell}$ is *cold* if and only if $\partial_1 a \cap \text{Trunk}(T, \partial\mathbf{v}) \neq \emptyset$.
- (ii) We say that a variable node $x \in V_{2\ell}$ is *cold* if $x \in \text{Trunk}(T, \partial\mathbf{v})$.
- (iii) We say that (x, a) (with $x \in \partial_1 a$) is *cold* if x is cold or a is cold.
- (iv) We say that a path $[x \rightarrow r]$ ($x \in \partial V_{2\ell}$) is *cold* if it contains at least $\lfloor 0.4\ell \rfloor$ cold pairs (x, a) .
- (v) Finally, we say that the pair $(T, \partial\mathbf{v}) \in \tilde{\mathcal{T}}_{2\ell} \times \mathcal{P}(\{-1, 1\})^{\partial V_{2\ell}}$ is *cold* if all the paths $[x \rightarrow r]$, with $x \in \partial V_{2\ell}$, are cold.

The key result of this section is the following estimate.

Proposition 3.4. *We have*

$$\mathbb{P}[(T, \partial \mathbf{v}) \text{ is cold}] \geq 1 - \ell^{-1}.$$

Then in Section 3.3 we shall prove the following.

Proposition 3.5. *If $(T, \partial \mathbf{v}) \in \mathcal{T}_{2\ell} \times \mathcal{P}(\{-1, 1\})^{\partial V_{2\ell}}$ is cold, then*

$$\|v_{T, \uparrow}^{\partial \mathbf{v}} - v_{T, \uparrow}^{(2\ell)}\|_{\infty} \leq \ell^{-1}.$$

Let us see how this implies Proposition 3.1.

Proof of Proposition 3.1. The first part of the proposition directly follows from the combination of Proposition 3.4 and Proposition 3.5. For the second part of the proposition, we first need to introduce one more notation. For $\hat{T} \in \widehat{\mathcal{T}}_{2\ell+1}$ and $j \in [k-1]$ we denote by $\hat{T}[j] \in \mathcal{T}_{2\ell}$ the subtree of \hat{T} pending below the j -th neighbor of the root. For a boundary condition $\partial \mathbf{v}$ over $\partial \hat{T}$, we denote by $\partial \mathbf{v}[j]$ its restriction to $\hat{T}[j]$.

We observe that if $\partial \mathbf{v}$ satisfies **H**, then so does $\partial \mathbf{v}[j]$. Moreover, for any $T \in \mathcal{T}_{2\ell}$ and $j \in [k-1]$, we have by definition of the Galton-Watson process

$$\mathbb{P}[\hat{T}[j] = T] \leq 2\mathbb{P}[T = T].$$

For $\hat{T} \in \widehat{\mathcal{T}}_{2\ell+1}$ and a boundary condition $\partial \mathbf{v}$, using (3.3) and Taylor's theorem, we get

$$\|\hat{\mathbf{v}}_{\hat{T}}^{\partial \mathbf{v}} - \hat{\mathbf{v}}_{\hat{T}}^{(2\ell+1)}\|_{\infty} \leq 8k \exp(2\beta) \sup_{j \in [k-1]} \|v_{\hat{T}[j]}^{\partial \mathbf{v}[j]} - v_{\hat{T}[j]}^{(2\ell)}\|_{\infty}.$$

Thereby, we obtain, using the previous observations

$$\begin{aligned} \mathbb{P}\left[\|\hat{\mathbf{v}}_{\hat{T}}^{\partial \mathbf{v}} - \hat{\mathbf{v}}_{\hat{T}}^{(2\ell+1)}\|_{\infty} \geq k^2 \exp(2\beta) \ell^{-1}\right] &\leq \mathbb{P}\left[\sup_{j \in [k-1]} \|v_{\hat{T}[j]}^{\partial \mathbf{v}[j]} - v_{\hat{T}[j]}^{(2\ell)}\|_{\infty} \geq \ell^{-1}\right] \\ &\leq (k-1) \mathbb{P}\left[\|v_{\hat{T}[1]}^{\partial \mathbf{v}[1]} - v_{\hat{T}[1]}^{(2\ell)}\|_{\infty} \geq \ell^{-1}\right] \\ &\leq 2(k-1) \mathbb{P}\left[\|v_T^{\partial \mathbf{v}[1]} - v_T^{(2\ell)}\|_{\infty} \geq \ell^{-1}\right]. \end{aligned}$$

For ℓ large enough, and using the first part of the proposition that we already proved, we have

$$\mathbb{P}\left[\|v_{T'}^{\partial \mathbf{v}[1]} - v_{T'}^{(2\ell)}\|_{\infty} \geq (\ell-1)^{-1}\right] \leq \ell^{-1}.$$

The second part of the proposition follows. □

Proof of Proposition 3.2. For $T \in \widetilde{\mathcal{T}}_{2\ell+2}$, let T' be the tree obtained by removing the tree pending below the last children of T 's root. Then if $(T', \partial \mathbf{v}'')$ is good, then so is $(T', \partial \mathbf{v}')$. In particular

$$\mathbb{P}[(T', \partial \mathbf{v}') \text{ is cold}] \geq \mathbb{P}[(T', \partial \mathbf{v}) \text{ is cold}] \geq 1 - \ell^{-1}.$$

In this case we have

$$\|v_{T'}^{\partial \mathbf{v}} - v_{T'}^{(2\ell+1)}\|_{\infty} \leq \ell^{-1}$$

and moreover, applying Taylor's theorem and by a similar token as previously,

$$\|\mu_T^{\partial \mathbf{v}'} - \mu_T^{(2\ell+2)}\|_{\infty} \leq k \exp(\beta) \ell^{-1}.$$

This concludes the proof of the proposition. □

3.3. Proof of Proposition 3.5. We begin with the following lemma, that shows that the messages exiting vertices $x \in \text{Trunk}(T, \partial v)$ are under tight control.

Lemma 3.6. *Let $(T, \partial v) \in \mathcal{T}_{2\ell} \times \mathcal{P}(\{-1, 1\})^{\partial V_{2\ell}}$ be fixed. For all $x \in \text{Trunk}(T, \partial v)$, we have*

$$v_{x, \uparrow}^{T, \partial v}(1) > 1 - \exp(-k\beta/2) \quad \text{and} \quad v_{x, \uparrow}^{T, \partial v^{(0)}}(1) > 1 - \exp(-k\beta/2).$$

The lemma will rely on a detailed analysis of terms of the form $\frac{\hat{v}_{x, \uparrow}^{T, \partial v}(1)}{\hat{v}_{x, \uparrow}^{T, \partial v}(-1)}$. In order to simplify the discussion, we shall isolate this analysis in the following lemma.

Lemma 3.7. *Let a clause a be fixed along with its adjacent variables $x \in \partial a$ and a family $(v_{x \rightarrow a})_{x \in \partial a} \in \mathcal{P}(\{-1, 1\})^k$. Let*

$$\partial_{\text{good}} a = \{x \in \partial a, v_{x \rightarrow a}(1) \geq 1 - \exp(-k\beta/2)\}.$$

Let $(\hat{v}_{a \rightarrow x})_{x \in \partial a} \in \mathcal{P}(\{-1, 1\})^k$ be defined by the following equations.

$$\hat{v}_{a \rightarrow x}(s) = \frac{\sum_{s_a \in \{-1, 1\}^k} \mathbf{1}_{s_x = s} \psi_{a, \beta}(s_a) \prod_{y \in \partial a \setminus \{x\}} v_{y \rightarrow a}(s_y)}{\sum_{s_a \in \{-1, 1\}^k} \psi_{a, \beta}(s_a) \prod_{y \in \partial a \setminus \{x\}} v_{y \rightarrow a}(s_y)} \quad (3.4)$$

Then for $x \in \partial a$, the following estimates hold true.

(a)

$$\exp(-\beta) \leq \frac{\hat{v}_{a \rightarrow x}(1)}{\hat{v}_{a \rightarrow x}(-1)} \leq \exp(\beta).$$

(b) *If $x \in \partial_1 a$, then*

$$\frac{\hat{v}_{a \rightarrow x}(1)}{\hat{v}_{a \rightarrow x}(-1)} \geq 1.$$

(c) *If $\{x\} = \partial_1 a$ and $\partial_{-1} a \subset \partial_{\text{good}} a$, then*

$$\frac{\hat{v}_{a \rightarrow x}(1)}{\hat{v}_{a \rightarrow x}(-1)} \geq \exp(0.99\beta).$$

(d) *If $|\partial_1(a) \setminus \{x\} \cap \partial_{\text{good}} a| \geq p$, then*

$$\frac{\hat{v}_{a \rightarrow x}(1)}{\hat{v}_{a \rightarrow x}(-1)} \geq \exp(-\exp(-pk\beta/3)).$$

Proof. Point (a) easily follows from the fact that for any $s_a \in \{-1, 1\}^{\partial a}$, $\exp(-\beta) \leq \psi_{a, \beta}(s_a) \leq 1$, and that if there is $x \in \partial a$ such that $s_x = 1$, then $\psi_{a, \beta}(s_a) = 1$.

Point (b) follows from the observation that if $s_a, s'_a \in \{-1, 1\}^{\partial a}$ satisfy $s_x = 1$ and $s_y = s'_y$ for $y \in \partial a \setminus \{x\}$, then $\psi_{a, \beta}(s_a) \geq \psi_{a, \beta}(s'_a)$.

Point (c) follows from the observation that, if $\{x\} = \partial_1 a$ and $\partial_{-1} a \subset \partial_{\text{good}} a$,

$$\exp(-\mathbf{1}_{s \neq 1} \beta) \leq \sum_{s_a \in \{-1, 1\}^{\partial a}} \mathbf{1}_{s_x = s} \psi_{a, \beta}(s_a) \prod_{y \in \partial a \setminus \{x\}} v_{y \rightarrow a}(s_y) \leq \exp(-\mathbf{1}_{s \neq 1} \beta) + 2k \exp(-k\beta/2).$$

Finally, point (d) is obtained by observing that, if $|\partial_1 a \setminus \{x\} \cap \partial_{\text{good}} a| \geq p$,

$$\left| \sum_{s_a \in \{-1, 1\}^{\partial a}} \mathbf{1}_{s_x = s} \psi_{a, \beta}(s_a) \prod_{y \in \partial a \setminus \{x\}} v_{y \rightarrow a}(s_y) - 1 \right| \leq \prod_{y \in \partial_1 a \setminus \{x\}} (1 - v_{y \rightarrow a}(1)) \leq \exp(-pk\beta/2).$$

□

Proof of Lemma 3.6. We first prove the statement concerning $v_{x, \uparrow}^{T, \partial v}$. We prove it by induction over $t = \ell - \frac{\text{dist}(x, r)}{2}$. For $t = 0$ the result holds by definition of $\text{Trunk}(T, \partial v)$. Assume that the results hold for all $x \in V_{2\ell}$ such that $\text{dist}(x, r) \geq 2(\ell - t)$ and let $x \in V_{2\ell}$ with $\text{dist}(x, r) = 2(\ell - t - 1)$ be fixed. We define

$$\Delta_1 x = \partial_1 x \cap \partial_1 x,$$

$$\Delta_{1,0} x = \{a \in \partial_1 x, \partial_{-1} a \cap \text{Trunk}(T, \partial v) = \partial_1 a\},$$

$$\Delta_{-1,0} x = \{a \in \partial_1 x, a \in \partial_{-1,0} x\},$$

$$\text{and for } 1 \leq l \leq k, \quad \Delta_{-1,l} x = \{a \in \partial_{-1}(x, a), |\partial_1(x)| = l, |\partial_1(x) \setminus \text{Trunk}(T, \partial v)| \leq |\partial_1(x)|/4\}$$

We have, by Eq. (3.2)

$$\frac{v_{x,\uparrow}^{T,\partial v}(1)}{v_{x,\uparrow}^{T,\partial v}(-1)} = \prod_{a \in \Delta_{1,0}x} \frac{\hat{v}_{a,\uparrow}^{T,\partial v}(1)}{\hat{v}_{a,\uparrow}^{T,\partial v}(-1)} \prod_{a \in \Delta_1 x \setminus \Delta_{1,0}x} \frac{\hat{v}_{a,\uparrow}^{T,\partial v}(1)}{\hat{v}_{a,\uparrow}^{T,\partial v}(-1)} \prod_{l=1}^k \prod_{a \in \Delta_{-1,l}x} \frac{\hat{v}_{a,\uparrow}^{T,\partial v}(1)}{\hat{v}_{a,\uparrow}^{T,\partial v}(-1)} \prod_{a \in \Delta_{-1,0}x} \frac{\hat{v}_{a,\uparrow}^{T,\partial v}(1)}{\hat{v}_{a,\uparrow}^{T,\partial v}(-1)} \prod_{a \in \partial_1 x \setminus (\Delta_1 x \cup (\cup_{l=0}^k \Delta_{-1,l}x))} \frac{\hat{v}_{a,\uparrow}^{T,\partial v}(1)}{\hat{v}_{a,\uparrow}^{T,\partial v}(-1)}. \quad (3.5)$$

Because the messages (v, \hat{v}) satisfy Eq. (3.4), we can apply the result of Lemma 3.7 to them. It follows that

$$\frac{v_{x,\uparrow}^{T,\partial v}(1)}{v_{x,\uparrow}^{T,\partial v}(-1)} \geq \exp(\beta[0.99|\Delta_{1,0}x| - |\Delta_{-1,0}x|]) \exp\left(\sum_{l=1}^k |\Delta_{-1,l}x| \exp(-k\beta l/4)\right) \exp\left(-2\beta|\partial_1 x \setminus (\Delta_1 x \cup (\cup_{l=0}^k \Delta_{-1,l}x))|\right).$$

By definition of $\text{Trunk}(T, \partial v)$, we have

$$|\Delta_{1,0}x| \geq \lceil 0.9k \rceil, \quad |\Delta_{-1,0}x| \leq \lceil 0.1k \rceil, \quad |\partial_1 x \setminus (\Delta_1 x \cup (\cup_{l=0}^k \Delta_{-1,l}x))| \leq k^{3/4}.$$

Moreover, for $1 \leq l \leq k$ we have $|\Delta_{-1,l}x| \leq |\partial_{-1,l}x| \leq k^{l+3}/l!$ (by **TR3**) and hereby

$$\sum_{l=1}^k |\Delta_{-1,l}x| \exp(-k\beta l/4) \leq \sum_{l=1}^k \frac{k^{l+3}}{l!} \exp(-k\beta l/4) \leq k^3 \exp(k \exp(-k\beta/4)) \leq 2.$$

Replacing with these four estimate in (3.5), we obtain $\frac{v_{x,\uparrow}^{T,\partial v}(1)}{v_{x,\uparrow}^{T,\partial v}(-1)} \geq \exp(2k\beta/3)$, as desired.

The second part of the lemma, regarding $v_{x,\uparrow}^{(2\ell)}$, follows from the observation that $\text{Trunk}(T, \partial v) \subset \text{Trunk}(T, \partial v^{(0)})$. \square

Proof of Proposition 3.5. Let $x \in V_{2\ell} \setminus \partial V_{2\ell}$ be fixed, and let for $a \in \partial_1 x$ and a boundary condition $\partial v, \hat{\varepsilon}_{a,\uparrow}^{\partial v} : \{-1, 1\} \rightarrow \mathbb{R}$ be defined by, for $s \in \{-1, 1\}$

$$\hat{\varepsilon}_{a,\uparrow}^{T,\partial v}(s) = \sum_{s_a \in \{-1, 1\}^{\partial a}} \mathbf{1}_{s_x = s} \psi_{a,\beta}(s_a) \prod_{y \in \partial_1 a} v_{y,\uparrow}^{T,\partial v}(s_y). \quad (3.6)$$

By applying Taylor's theorem to equation (3.2), observing that $v_{x,\uparrow}^{T,\partial v}(1) = \left(1 + \prod_{a \in \partial_1 x} \frac{\hat{\varepsilon}_{a,\uparrow}^{T,\partial v}(-1)}{\hat{\varepsilon}_{a,\uparrow}^{T,\partial v}(1)}\right)^{-1}$ we obtain

$$\|v_{x,\uparrow}^{T,\partial v} - v_{x,\uparrow}^{T,\partial v(0)}\|_\infty \leq \sup_{u \in [0,1]} \frac{\left(\frac{uv_{x,\uparrow}^{T,\partial v}(-1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(-1)}{uv_{x,\uparrow}^{T,\partial v}(1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(1)}\right)}{\left(1 + \frac{uv_{x,\uparrow}^{T,\partial v}(-1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(-1)}{uv_{x,\uparrow}^{T,\partial v}(1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(1)}\right)^2} \sum_{a \in \partial_1 x} \sup_{u \in [0,1]} \left\| \frac{\hat{\varepsilon}_{a,\uparrow}^{T,\partial v} - \hat{\varepsilon}_{a,\uparrow}^{T,\partial v(0)}}{u\hat{\varepsilon}_{a,\uparrow}^{T,\partial v} + (1-u)\hat{\varepsilon}_{a,\uparrow}^{T,\partial v(0)}} \right\|_\infty. \quad (3.7)$$

We observe that for $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ we have

$$\frac{(x_1 + x_2 + x_3)}{(1 + x_1 + x_2 + x_3)^2} \leq \min\{x_1 + x_2 + x_3, (x_1 + x_2 + x_3)^{-1}\} \leq 1, \\ \sup_{u \in [0,1]} \frac{ux_1 + (1-u)x_2}{ux_3 + (1-u)x_4} \leq \frac{\max\{x_1, x_2\}}{\min\{x_3, x_4\}}.$$

Using this in conjunction with Lemma 3.6 we obtain

$$\sup_{u \in [0,1]} \frac{\left(\frac{uv_{x,\uparrow}^{T,\partial v}(-1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(-1)}{uv_{x,\uparrow}^{T,\partial v}(1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(1)}\right)}{\left(1 + \frac{uv_{x,\uparrow}^{T,\partial v}(-1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(-1)}{uv_{x,\uparrow}^{T,\partial v}(1) + (1-u)v_{x,\uparrow}^{T,\partial v(0)}(1)}\right)^2} \leq 6 \exp(-k\beta \mathbf{1}_{x \text{ is cold}}/2). \quad (3.8)$$

We further observe that

$$\sup_{u \in [0,1]} \left\| \frac{\hat{\varepsilon}_{a,\uparrow}^{T,\partial v} - \hat{\varepsilon}_{a,\uparrow}^{T,\partial v^{(0)}}}{u \hat{\varepsilon}_{a,\uparrow}^{T,\partial v} + (1-u) \hat{\varepsilon}_{a,\uparrow}^{T,\partial v^{(0)}}} \right\|_{\infty} \leq \exp(\beta) \left\| \hat{\varepsilon}_{a,\uparrow}^{T,\partial v} - \hat{\varepsilon}_{a,\uparrow}^{T,\partial v^{(0)}} \right\|_{\infty}.$$

In particular,

$$\|v_{x,\uparrow}^{T,\partial v} - v_{x,\uparrow}^{T,\partial v^{(0)}}\|_{\infty} \leq 6 \exp(\beta) \exp(-k\beta \mathbf{1}_{x \text{ is cold}}/2).$$

Using again Taylor's theorem, for any $a \in F_{2\ell}$ we have

$$\left\| \hat{\varepsilon}_{a,\uparrow}^{T,\partial v} - \hat{\varepsilon}_{a,\uparrow}^{T,\partial v^{(0)}} \right\|_{\infty} \leq 4 \exp(-k\beta/2) \sum_{z \in \partial_1 a} \min_{\substack{y \in \partial_1 a \cap \partial_1 a \\ y \neq z}} \max \left\{ v_{y,\uparrow}^{T,\partial v}(-1), v_{y,\uparrow}^{T,\partial v^{(0)}}(-1) \right\} \|v_{z,\uparrow}^{T,\partial v} - v_{z,\uparrow}^{T,\partial v^{(0)}}\|_{\infty}. \quad (3.9)$$

Therefore, if a is cold and the unique x such that $\delta_1 a = \{x\}$ is not cold, we have

$$\left\| \hat{\varepsilon}_{a,\uparrow}^{T,\partial v} - \hat{\varepsilon}_{a,\uparrow}^{T,\partial v^{(0)}} \right\|_{\infty} \leq \exp(-k\beta/2) \sum_{x \in \partial_1 a} \mathbf{1}_{x \text{ not cold}} \|v_{x,\uparrow}^{T,\partial v} - v_{x,\uparrow}^{T,\partial v^{(0)}}\|_{\infty} + \sum_{x \in \partial_1 a} \mathbf{1}_{x \text{ is cold}} \|v_{x,\uparrow}^{T,\partial v} - v_{x,\uparrow}^{T,\partial v^{(0)}}\|_{\infty}. \quad (3.10)$$

Combining (3.8) with (3.10), we obtain

$$\left\| v_{x,\uparrow}^{T,\partial v} - v_{x,\uparrow}^{T,\partial v^{(0)}} \right\|_{\infty} \leq 24 \exp(\beta) \sum_{a \in \partial_1 x} \sum_{y \in \partial_1 a} \exp(-k\beta(\mathbf{1}_{x \text{ is cold}} + \mathbf{1}_{x \text{ is not cold}} \mathbf{1}_a \text{ is cold} \mathbf{1}_y \text{ is not cold})/2) \|v_{y,\uparrow}^{T,\partial v} - v_{y,\uparrow}^{T,\partial v^{(0)}}\|_{\infty}.$$

Iterating this equation, we obtain, (using that for any $x \in \partial V_{2\ell}$, the path from the root r of T to x contains at least $\lfloor 0.4\ell \rfloor$ cold pairs (x, a))

$$\begin{aligned} \left\| v_{x,\uparrow}^{\partial v} - v_{x,\uparrow}^{(2\ell)} \right\|_{\infty} &\leq 24^\ell \exp(\beta\ell) \sum_{x \in \partial V_{2\ell}} \exp(-k\beta \lfloor 0.4\ell \rfloor / 2) \|\partial v_x - \partial v_x^{(0)}\|_{\infty} \\ &\leq |\partial V_{2\ell}| 24^\ell \exp(\beta\ell) \exp(-k\beta \lfloor 0.4\ell \rfloor / 2) \\ &\leq (dk)^\ell 24^\ell \exp(-0.01k^2\ell) = o_\ell(1). \end{aligned}$$

□

3.4. Proof of Proposition 3.4.

Proof. In order to prove the proposition, we will need to slightly extend the notion of cold variables and cold clauses. Let a pair $(T, \partial v) \in \mathcal{T}_{2\ell} \times \mathcal{P}(\{-1, 1\})^{d_\ell}$ be fixed. Given $v \in T$ and $\{w\} = \partial_1 v$, let T_v denote the subtree of $T \setminus \{w\}$ rooted at v . Also recall that we denoted by $\partial V_{2\ell} = \partial T$.

For (a, x) with $\{x\} = \partial_1 a$, we say that x is strongly cold with respect to a for the pair $(T, \partial v)$ if there exists no tree $T' \in \mathcal{T}_{2\ell}$ and no boundary condition $\partial v'$ over $\partial V_{2\ell}$ such that the following is true.

- x is not cold in $(T', \partial v')$,
- $T'_x = T_x$,
- $\forall x \in \partial T_x \setminus \partial T_a, (\partial v')_x = (\partial v)_x$.

Observe that strongly cold variables are also cold. Let, for $a' \in \partial_1 x$, $p_{x,a'}$ be the probability that x is not strongly cold with respect to a' when the pair $(T, \partial v)$ is drawn at random. We shall prove by induction over $t \in \{1, \dots, \ell\}$ that for x at distance $2t$ from $\partial V_{2\ell}$ and $a' \in \partial_1 x$, $p_{x,a'} \leq 2^{-0.9k}$. For $t = 0$, the result follows from the assumption on the distribution of ∂v . Let us now assume that we have proved the result up to $t \geq 0$ and consider x at distance $2(t+1)$ from $\partial V_{2\ell}$ and $a' \in \partial_1 x$. For x not to be strongly cold with respect to a' , one of the following must happen. Let $V_{\text{cold}}^{(t)}$ be the set of strongly cold variables at distance t from $\partial V_{2\ell}$. Let a be the clause such that $\partial_1 x = \{a\}$.

- there are less than $\lfloor 0.95k \rfloor$ clauses $a \in \partial_1 x$ such that $\partial_1 a = \{x\}$,
- there are more than $\lceil 0.05k \rceil$ clauses $a \in \partial x$ such that $|\partial_{-1} a| = k$,
- there is $1 \leq l \leq k$ such that there are more than $0.5k^{l+3}/l!$ clauses $a \in \partial_{-1} x$ with $|\partial_1 a| = l$,
- $\left| \{b \in \partial_{1,0} x \setminus \{a\}, \partial b \setminus \{x\} \notin V_{\text{cold}}^{(t-1)}\} \right| \geq k^{3/4} - 1$,
- $|\{b \in \partial_{-1} x, |\partial_{-1} b| \leq k, |\partial_1 b \setminus \{x\} \setminus V_{\text{cold}}^{(t-1)}| \geq |\partial_1 b|/4\}| \geq k^{3/4} - 1$.

By definition of our random process, (a), (b) and (c) each hold with probability at most $2^{-0.95k}$. The probability that a given clause $b \in \partial_{1,0}x \setminus \{a\}$ contains at least one not strongly cold variable different from x is $\sum_{y \in \partial_{1,0}x \setminus \{a\}} p_{y,b} + \tilde{O}_k(2^{-k}) \leq k2^{-0.9k}$. By definition, the probability that each of the clauses $b_1, \dots, b_y \in \partial_{1,0}x \setminus \{a\}$ contain at least one not strongly cold variable different from x is upperbounded by $\binom{|\partial_{1,0}x \setminus \{a\}|}{y} (k2^{-0.9k})^y$. Therefore, the probability that (d) happens is at most $2^{-1.5k}$. Similarly, (e) happen with probability at most $2^{-1.5k}$. Therefore we obtain $p_{x,a'} \leq 3 \cdot 2^{-0.95k} + 2 \cdot 2^{-1.5k} \leq 2^{-0.9k}$, as needed.

Let a clause $a \in F_{2\ell}$ be fixed as well as $x \in \partial_{\downarrow} a$. Let $\partial_{\uparrow} a = \{y\}$. We say that a is strongly cold with respect to x if $|\partial_{\downarrow} a \setminus \{x, y\} \cap V_{\text{cold}}^{(t)}| \geq 1$. Let $q_{a,x}$ denote the probability that a is not strongly cold with respect to $x \in \partial_{\downarrow} a$ when the pair $(T, \partial \mathbf{v})$ is drawn from a distribution that satisfies the hypothesis of the proposition. By our previous estimate (g) and (h) happen with probability at most $2^{-1.5k}$ while (f) happens with probability at most $2^{-0.9k}$. In particular, $q_{a,x} \leq 2^{-0.9k}$. By construction of the strongly cold clauses, the probability that a pair (x, a) with $\{a\} = \partial_{\uparrow} x$ is not strongly cold with respect to $b \in \partial_{\downarrow} x$ is then upperbounded by $p_{x,b} q_{a,x} \leq 2^{-1.7k}$.

Let $x \in \partial V_{2\ell}$ be fixed. Recall that we denoted by r the root of T . We denote sequence of variables and clauses on the path $[x \rightarrow r]$ by $(x_0 = x, a_0, x_1, a_1, \dots, r)$. For the path $[x \rightarrow r]$ not to be cold, there must be at least $\lfloor 0.6\ell \rfloor$ pairs (x_i, a_i) along this path that are not strongly cold. Moreover, for $i_1 \neq i_2 \neq \dots \neq i_l$, the probability that the (x_{i_j}, a_{i_j}) are strongly cold is independent (by construction). Therefore we obtain

$$\begin{aligned} \mathbb{P}[[x \rightarrow r] \text{ not cold}] &\leq \sum_{l \geq \lfloor 0.6\ell \rfloor} \sum_{0 \leq i_1 \leq \dots \leq i_l < \ell} \mathbb{P}[(x_{i_1}, a_{i_1}) \text{ not strongly cold and } (x_{i_2}, a_{i_2}) \text{ not strongly cold and} \\ &\quad \dots \text{ and } (x_{i_l}, a_{i_l}) \text{ not strongly cold}] \\ &\leq \sum_{l \geq \lfloor 0.6\ell \rfloor} \sum_{0 \leq i_1 \leq \dots \leq i_l < \ell} \prod_{j=1}^l \mathbb{P}[(x_{i_j}, a_{i_j}) \text{ not strongly cold}] \\ &\leq 2^\ell \left(2^{-1.7k}\right)^{\lfloor 0.6\ell \rfloor} \leq 2^{-1.02k\ell}. \end{aligned}$$

Consequently, we obtain with the union bound

$$\mathbb{P}[(T, \partial \mathbf{v}) \text{ is not cold}] \leq \sum_{x \in \partial V_{2\ell}} \mathbb{P}[[x \rightarrow r] \text{ not cold}] \leq |\partial V_{2\ell}| 2^{-1.02k\ell} \leq (dk)^\ell 2^{-1.02k\ell} = o_\ell(1).$$

□

4. THE FIXED POINT PROBLEM ON TREES

In this section we prove Proposition 1.2 and Lemma 2.3.

We shall obtain the propositions by making the connection between the skewed fixed points of the operator $\mathcal{G}_{k,d,\beta}$ and the analysis of Belief Propagation on random Galton-Watson trees studied in the previous section. We first identify $\mathcal{P}(\{-1, 1\})$ with $(0, 1)$ through $\eta \mapsto \eta(-1)$. This also identifies $\mathcal{P}(\mathcal{P}(\{-1, 1\}))$ with $\mathcal{P}(0, 1)$. With the notations of the previous section (and, in particular, q given by (3.1)), we shall prove that

Proposition 4.1. *Let π be a skewed fixed point of $\mathcal{G}_{k,d,\beta}$ and $\ell \geq 1$ be fixed. Then we have*

$$\pi = \sum_{T \in \mathcal{T}_{2\ell}} p_{k,d,\beta}^{(2\ell)}(T) \int_{\mathcal{P}(\{-1, 1\})^{\partial V_{2\ell}}} \delta_{\mathbf{v}_T^{\partial \mathbf{v}}} \bigotimes_{x \in \partial V_{2\ell}} \left(\mathbf{1}_{b_{x,1}=-1} \frac{1 - \partial \mathbf{v}_x(-1)}{1 - q} d\pi(\partial \mathbf{v}_x) + \mathbf{1}_{b_{x,1}=1} \frac{\partial \mathbf{v}_x(-1)}{q} d\pi(\partial \mathbf{v}_x) \right).$$

Let us see how Proposition 1.2 follows from this proposition and from the result of the previous section.

Proof of Proposition 1.2. Let, for $\ell \geq 0$, $\pi^{(2\ell)} \in \mathcal{P}(\{-1, 1\})$ be the distribution of $\mathbf{v}_T^{(2\ell)}$. Let \mathbf{v} be distributed according to π . It follows from Proposition 4.1 that $\mathbf{v} = \mathbf{v}_T^{\partial \mathbf{v}}$, where T and $\partial \mathbf{v}$ satisfies the assumptions of Section 3. In particular it follows from Proposition 3.1 that π weakly converges towards $\pi^{(2\ell)}$, hence the unicity of the fixed point. By a similar reasoning, we see that $\pi^{(2\ell)}$ admits a weak limit, proving the existence of the fixed point. □

4.1. The multi-type Galton-Watson branching process: proof of Proposition 4.1. For $\pi, \hat{\pi} \in \mathcal{P}(0, 1)$ we define

$$h(\pi) = \int_{(0,1)} \eta d\pi(\eta), \quad \hat{h}(\hat{\pi}) = \int_{(0,1)} \hat{\eta} d\hat{\pi}(\hat{\eta}).$$

We let $f : (0, 1)^{d-1} \rightarrow (0, 1)$ (resp. $\hat{f} : (0, 1)^{k-1} \rightarrow (0, 1)$) be defined by

$$f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1}) = \frac{\prod_{j=1}^{d/2-1} \hat{\eta}_j \prod_{j=d/2}^{d-1} (1 - \hat{\eta}_j)}{z(\hat{\eta}_1, \dots, \hat{\eta}_{d-1})}, \quad \hat{f}(\eta_1, \dots, \eta_{k-1}) = \frac{1 - c_\beta \prod_{i=1}^{k-1} \eta_i}{\hat{z}(\eta_1, \dots, \eta_{k-1})},$$

and $f_d, \hat{f}_{k,\beta} : [0, 1] \rightarrow (0, 1)$ be defined by

$$f_d(\hat{\eta}) = f(\hat{\eta}, \dots, \hat{\eta}) = 1 - \hat{\eta}, \quad \hat{f}_{k,\beta}(\eta) = \hat{f}(\eta, \dots, \eta).$$

We say that $(\pi, \hat{\pi})$ is a fixed point of $(\mathcal{F}_{k,d,\beta}, \widehat{\mathcal{F}}_{k,d,\beta})$ iff $\pi = \mathcal{F}_{k,d,\beta}(\hat{\pi})$ and $\hat{\pi} = \widehat{\mathcal{F}}_{k,d,\beta}(\pi)$.

Lemma 4.2. *If $(\pi, \hat{\pi})$ is a fixed point of $(\mathcal{F}_{k,d,\beta}, \widehat{\mathcal{F}}_{k,d,\beta})$, then we have*

$$h[\pi] = f_d(\hat{h}[\hat{\pi}]), \quad \hat{h}[\hat{\pi}] = \hat{f}_{k,\beta}(h[\pi]).$$

Proof. We first observe that, using the multilinearity of z (resp. \hat{z})

$$\begin{aligned} Z[\hat{\pi}] &= \int_{(0,1)^{d-1}} z(\hat{\eta}_1, \dots, \hat{\eta}_{d-1}) \bigotimes_{j=1}^{d-1} d\hat{\pi}(\hat{\eta}_j) = z(\hat{h}[\hat{\pi}], \dots, \hat{h}[\hat{\pi}]), \\ \hat{Z}[\pi] &= \int_{(0,1)^{k-1}} \hat{z}(\eta_1, \dots, \eta_{k-1}) \bigotimes_{j=1}^{k-1} d\pi(\eta_j) = \hat{z}(h[\pi], \dots, h[\pi]). \end{aligned}$$

Using these equations, we obtain

$$\begin{aligned} h[\pi] &= \int_{(0,1)} \eta d\mathcal{F}_{d,k,\beta}[\hat{\pi}](\eta) = \frac{1}{Z[\hat{\pi}]} \int_{(0,1)^{d-1}} z(\hat{\eta}_1, \dots, \hat{\eta}_{d-1}) f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1}) \bigotimes_{j=1}^{d-1} d\hat{\pi}(\hat{\eta}_j) \\ &= \frac{1}{Z[\hat{\pi}]} \int_{(0,1)^{d-1}} \left[\prod_{j=1}^{d/2-1} \hat{\eta}_j \prod_{j=d/2}^{d-1} (1 - \hat{\eta}_j) \right] \bigotimes_{j=1}^{d-1} d\hat{\pi}(\hat{\eta}_j) \\ &= f_d(\hat{h}[\hat{\pi}]). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \hat{h}[\hat{\pi}] &= \int_{(0,1)} \hat{\eta} d\widehat{\mathcal{F}}_{d,k,\beta}[\pi](\hat{\eta}) = \frac{1}{\hat{Z}[\pi]} \int_{(0,1)^{k-1}} \hat{z}(\eta_1, \dots, \eta_{k-1}) f(\eta_1, \dots, \eta_{k-1}) \bigotimes_{j=1}^{k-1} d\pi(\eta_j) \\ &= \frac{1}{\hat{Z}[\pi]} \int_{(0,1)^{k-1}} \left[1 - c_\beta \prod_{j=1}^{k-1} \eta_j \right] \bigotimes_{j=1}^{k-1} d\pi(\eta_j) \\ &= \hat{f}_{k,\beta}(h[\pi]). \end{aligned}$$

□

Recalling the definition of $q = q(d, k, \beta)$ in Eq. (3.1), and defining $\hat{q} = 1 - q$, the following is a simple observation.

Fact 4.3. *The set of equations*

$$y = 1 - \hat{y}, \quad \hat{y} = \hat{f}_{k,\beta}(y),$$

admits for unique solution in $[0, 1]^2$ the pair (q, \hat{q}) .

We define the measures $\pi_+, \pi_-, \hat{\pi}_+$ and $\hat{\pi}_-$ over $(0, 1)$ by

$$d\pi_+(\eta) = \frac{1-\eta}{1-q} d\pi(\eta), \quad d\pi_-(\eta) = \frac{\eta}{q} d\pi(\eta), \quad (4.1)$$

$$d\hat{\pi}_+(\hat{\eta}) = \frac{1-\hat{\eta}}{1-\hat{q}} d\hat{\pi}(\hat{\eta}), \quad d\hat{\pi}_-(\hat{\eta}) = \frac{\hat{\eta}}{\hat{q}} d\hat{\pi}(\hat{\eta}). \quad (4.2)$$

Lemma 4.4. *If $(\pi, \hat{\pi})$ is a fixed point of $(\mathcal{F}_{k,d,\beta}, \widehat{\mathcal{F}}_{k,d,\beta})$, we have*

$$\pi_- = \int_{(0,1)^{d-1}} \delta_{f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1})} \bigotimes_{j=1}^{d/2-1} d\hat{\pi}_-(\hat{\eta}_j) \bigotimes_{j=d/2}^{d-1} d\hat{\pi}_+(\hat{\eta}_j), \quad (4.3)$$

$$\pi_+ = \int_{(0,1)^{d-1}} \delta_{f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1})} \bigotimes_{j=1}^{d/2-1} d\hat{\pi}_+(\hat{\eta}_j) \bigotimes_{j=d/2}^{d-1} d\hat{\pi}_-(\hat{\eta}_j), \quad (4.4)$$

$$\begin{aligned} \hat{\pi}_- &= \sum_{r=1}^{k-1} \binom{k-1}{r} \frac{q^r (1-q)^{k-1-r}}{1-c_\beta q^{k-1}} \\ &\quad \int_{(0,1)^{k-1}} \delta_{\hat{f}(\eta_1, \dots, \eta_{k-1})} \bigotimes_{j=1}^r d\pi_-(\eta_j) \bigotimes_{j=r+1}^{k-1} d\pi_+(\eta_j) \end{aligned} \quad (4.5)$$

$$\begin{aligned} &+ \exp(-\beta) \frac{q^{k-1}}{1-c_\beta q^{k-1}} \int_{(0,1)^{k-1}} \delta_{\hat{f}(\eta_1, \dots, \eta_{k-1})} \bigotimes_{j=1}^{k-1} d\pi_-(\eta_j), \\ \hat{\pi}_+ &= \sum_{r=0}^{k-1} \binom{k-1}{r} q^r (1-q)^{k-1-r} \\ &\quad \int_{(0,1)^{k-1}} \delta_{\hat{f}(\eta_1, \dots, \eta_{k-1})} \bigotimes_{j=1}^r d\pi_-(\eta_j) \bigotimes_{j=r+1}^{k-1} d\pi_+(\eta_j). \end{aligned} \quad (4.6)$$

Proof. We first observe that, recalling the definition of z in Section 1.2, $qZ[\hat{\pi}] = \hat{q}^{d/2-1}(1-\hat{q})^{d/2}$. We then compute

$$\begin{aligned} \pi_- &= \int_{(0,1)} \frac{\eta}{q} \delta_\eta d\pi(\eta) = \int_{(0,1)} \frac{\eta}{q} \delta_\eta d\mathcal{F}_{k,d,\beta}[\hat{\pi}](\eta) \\ &= \frac{1}{Z[\hat{\pi}]} \int_{(0,1)^{d-1}} \frac{1}{q} z(\hat{\eta}_1, \dots, \hat{\eta}_{d-1}) f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1}) \delta_{f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1})} \bigotimes_{j=1}^{d-1} d\hat{\pi}(\hat{\eta}_j) \\ &= \int_{(0,1)^{d-1}} \frac{\prod_{j=1}^{d/2-1} \hat{\eta}_j \prod_{j=d/2}^{d-1} (1-\hat{\eta}_j)}{\prod_{j=1}^{d/2-1} \hat{q} \prod_{j=d/2}^{d-1} (1-\hat{q})} \delta_{f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1})} \bigotimes_{j=1}^{d-1} d\hat{\pi}(\hat{\eta}_j) \\ &= \int_{(0,1)^{d-1}} \delta_{f(\hat{\eta}_1, \dots, \hat{\eta}_{d-1})} \bigotimes_{j=1}^{d/2-1} d\hat{\pi}_-(\hat{\eta}_j) \bigotimes_{j=d/2}^{d-1} d\hat{\pi}_+(\hat{\eta}_j) \end{aligned}$$

The equation on π_+ is proved similarly. We also compute

$$\begin{aligned} \hat{\pi}_- &= \int_{(0,1)} \frac{\hat{\eta}}{\hat{q}} \delta_{\hat{\eta}} d\hat{\pi}(\hat{\eta}) = \int_{(0,1)} \frac{\hat{\eta}}{\hat{q}} \delta_{\hat{\eta}} d\widehat{\mathcal{F}}_{k,d,\beta}[\pi](\hat{\eta}) \\ &= \int_{(0,1)^{k-1}} \frac{1-c_\beta \prod \eta_j}{1-c_\beta q^{k-1}} \delta_{\hat{f}(\eta_1, \dots, \eta_{k-1})} \bigotimes_{j=1}^{k-1} d\pi(\eta_j) \\ &= \sum_{r=0}^{k-2} \binom{k-1}{r} \frac{q^r (1-q)^{k-1-r}}{1-c_\beta q^{k-1}} \\ &\quad \int_{(0,1)^{k-1}} \delta_{\hat{f}(\eta_1, \dots, \eta_{k-1})} \bigotimes_{j=1}^r d\pi_-(\eta_j) \bigotimes_{j=r+1}^{k-1} d\pi_+(\eta_j) \\ &\quad + \exp(-\beta) \frac{q^{k-1}}{1-c_\beta q^{k-1}} \int_{(0,1)^{k-1}} \delta_{\hat{f}(\eta_1, \dots, \eta_{k-1})} \bigotimes_{j=1}^{k-1} d\pi_-(\eta_j). \end{aligned}$$

The equation on $\hat{\pi}_+$ is proved in a similar manner. □

Proof of Proposition 4.1. We observe that $\pi = (1-q)\pi_+ + q\pi_-$. Replacing with Lemma 4.4 yields

$$\pi = \sum_{T \in \mathcal{T}_2} p_{k,d,\beta}^{(2)}(T) \int_{\mathcal{P}(\{-1,1\})^{\partial V_2}} \delta_{v_T^{\partial V}} \bigotimes_{x \in \partial V_2} \left(\mathbf{1}_{b_{x,1}=-1} \frac{1-\partial v_x(-1)}{1-q} d\pi(\partial v_x) + \mathbf{1}_{b_{x,1}=1} \frac{\partial v_x(-1)}{q} d\pi(\partial v_x) \right).$$

By induction over $1 \leq t \leq \ell$, using repeatedly Lemma 4.4, we obtain that

$$\pi = \sum_{T \in \mathcal{T}_{2t}} p_{k,d,\beta}^{(2t)}(T) \int_{\mathcal{P}(\{-1,1\})^{\partial V_{2t}}} \delta_{v_T^{\partial V}} \bigotimes_{x \in \partial V_{2t}} \left(\mathbf{1}_{b_{x,1}=-1} \frac{1-\partial v_x(-1)}{1-q} d\pi(\partial v_x) + \mathbf{1}_{b_{x,1}=1} \frac{\partial v_x(-1)}{q} d\pi(\partial v_x) \right).$$

This concludes the proof of the proposition. □

We define, for $(v_1, \dots, v_k, \hat{v}_1, \dots, \hat{v}_d) \in \mathcal{P}(\{-1, 1\})^{k+d}$ and $(b_1, \dots, b_k) \in \{-1, 1\}^k$,

$$\begin{aligned} z_1(\hat{v}_1, \dots, \hat{v}_d) &= \prod_{j \leq d/2} v_j(-1) \prod_{j > d/2} v_j(1) + \prod_{j \leq d/2} v_j(1) \prod_{j > d/2} v_j(-1), \\ z_2(v_1, \dots, v_k, b_1, \dots, b_k) &= 1 - c_\beta \prod_{j=1}^k v_j(b_j), \\ z_3(v_1, \hat{v}_1) &= v_1(1)\hat{v}_1(1) + v_1(-1)\hat{v}_1(-1). \end{aligned}$$

In order to prove Lemma 2.3, we also recall the following standard result, which we prove in Section 7.

Proposition 4.5. *We have*

$$\frac{1}{n} \ln \mathbb{E}[Z_\beta(\Phi)] \sim \ln 2 + \frac{d}{k} \ln(1 - c_\beta q^k) - \frac{d}{2} \ln\left(\frac{1}{2q}\right) - \frac{d}{2} \ln\left(\frac{1}{2(1-q)}\right).$$

Proof of Lemma 2.3. We compute

$$\begin{aligned} \ln \mathbb{E}[z_1(\hat{v}_1, \dots, \hat{v}_d)] &= \ln(2q^{d/2}(1-q)^{d/2}), \\ \ln \mathbb{E}[z_2(v_1, \dots, v_k, b_1, \dots, b_k)] &= \ln(1 - c_\beta q^k), \\ \ln \mathbb{E}[z_3(v_1, \hat{v}_1)] &= \ln(2q(1-q)). \end{aligned}$$

Thereby we have

$$\mathcal{F}(k, d, \beta) = \ln 2 + \frac{d}{k} \ln(1 - c_\beta q^k) - \frac{d}{2} \ln\left(\frac{1}{2q}\right) - \frac{d}{2} \ln\left(\frac{1}{2(1-q)}\right).$$

The proposition then follows from Proposition 4.5. \square

4.2. Finite ℓ approximations of $\mathcal{B}(k, d, \beta)$. We finally present a simple approximation of $\mathcal{B}(k, d, \beta)$ that will be useful in the following. Recall that $\text{GW}(k, d, \beta, 2\ell)$ and $\text{GW}'(k, d, \beta, 2\ell)$ were defined in the previous section. Let $\hat{p}_{k,d,\beta}^{(2\ell+1)}(\hat{T}_1, \dots, \hat{T}_d)$ denote the probability that the neighborhood of the root in is equal to $(\hat{T}_1, \dots, \hat{T}_d)$ under the process $\text{GW}'(k, d, \beta, 2\ell + 2)$. Denoting by e_1 the first edge exiting the root of the random process $\text{GW}'(k, d, \beta, 2\ell + 2)$, let for $T \in \mathcal{T}_{2\ell}$ and $\hat{T} \in \widehat{\mathcal{T}}_{2\ell+1}$ $\check{p}(T, \hat{T})$ be the probability that the 2ℓ -neighborhood (resp. $2\ell + 1$ -neighborhood) of this edge when removing its clause node (resp. variable node) is formed of the tree T (resp. \hat{T}). Similarly, denoting by a_1 the first clause connected to the root of the random process $\text{GW}'(k, d, \beta, 2\ell + 2)$, let for $T_1, \dots, T_k \in \mathcal{T}_\ell$ $\hat{p}_{k,d,\beta}(T_1, \dots, T_k)$ be the probability that the 2ℓ -neighborhood of a_1 is equal to (T_1, \dots, T_k) under the process $\text{GW}'(k, d, \beta, 2\ell + 2)$.

Finally, let $\pi_{k,d,\beta}^*$ be the unique skewed fixed point of $\mathcal{G}_{k,d,\beta}$ and, for the ease of notations, let $\pi_+, \pi_-, \hat{\pi}_+, \hat{\pi}_-$ denote the quantities associated to $\pi_{k,d,\beta}^*$ through Eq. (4.1-4.2).

Lemma 4.6. *We have*

$$\begin{aligned} \mathcal{B}(k, d, \beta) &= \sum_{(\hat{T}_1, \dots, \hat{T}_d) \in \widehat{\mathcal{T}}_{2\ell+1}^d} p_{k,d,\beta}^{(2\ell+1)}(\hat{T}_1, \dots, \hat{T}_d) \int_{(0,1)^d} \ln[z_1(\hat{v}_{\hat{T}_1}^{\partial v_1}, \dots, \hat{v}_{\hat{T}_d}^{\partial v_d})] \bigotimes_{j=1}^d \bigotimes_{x \in \partial_1 \hat{T}_j} d\pi_+((\partial v_j)_x) \bigotimes_{x \in \partial_{-1} \hat{T}_j} d\pi_-((\partial v_j)_x) \\ &\quad + \frac{d}{k} \sum_{(T_1, \dots, T_k) \in \mathcal{T}_{2\ell}^k} \hat{p}_{k,d,\beta}^{(2\ell)}(T_1, \dots, T_k) \int_{(0,1)^k} \ln[z_2(v_{T_1}^{\partial v_1}, \dots, v_{T_k}^{\partial v_k})] \bigotimes_{j=1}^k \bigotimes_{x \in \partial_1 T_j} d\pi_+((\partial v_j)_x) \bigotimes_{x \in \partial_{-1} T_j} d\pi_-((\partial v_j)_x) \\ &\quad - \sum_{T \in \mathcal{T}_{2\ell}, \hat{T} \in \widehat{\mathcal{T}}_{2\ell+1}} \check{p}_{k,d,\beta}^{(2\ell+1)}(T, \hat{T}) \int_{(0,1)^2} \ln[z_3(v_T^{\partial v_1}, \hat{v}_{\hat{T}}^{\partial v_2})] \bigotimes_{j=1}^2 \bigotimes_{x \in \partial_1 \hat{T}_j} d\pi_+((\partial v_j)_x) \bigotimes_{x \in \partial_{-1} \hat{T}_j} d\pi_-((\partial v_j)_x) + o_\ell(1). \end{aligned}$$

Proof. The proof is obtained by writing the expectation values in the definition of $\mathcal{B}(k, d, \beta)$ explicitly in terms of $\pi_+, \pi_-, \hat{\pi}_+, \hat{\pi}_-$ and by following steps similar to the one of the proof of Proposition 4.1. \square

We define

$$\begin{aligned}\mathcal{B}^{(\ell)}(k, d, \beta) &= \sum_{(\hat{T}_1, \dots, \hat{T}_d) \in \widehat{\mathcal{T}}_{2\ell+1}^d} \hat{p}_{k,d,\beta}^{(2\ell+1)}(\hat{T}_1, \dots, \hat{T}_d) \ln \left[z_1(\hat{v}_{\hat{T}_1}^{(2\ell+1)}, \dots, \hat{v}_{\hat{T}_d}^{(2\ell+1)}) \right] \\ &\quad + \frac{d}{k} \sum_{(T_1, \dots, T_k) \in \mathcal{T}_{2\ell}^k} p_{k,d,\beta}^{(2\ell)}(T_1, \dots, T_k) \ln \left[z_2(v_{T_1}^{(2\ell)}, \dots, v_{T_k}^{(2\ell)}) \right] \\ &\quad - \sum_{T \in \mathcal{T}_{2\ell}, \hat{T} \in \widehat{\mathcal{T}}_{2\ell+1}} \check{p}_{k,d,\beta}^{(2\ell+1)}(T, \hat{T}) \ln \left[z_3(v_T^{(2\ell)}, \hat{v}_{\hat{T}}^{(2\ell+1)}) \right].\end{aligned}$$

Proposition 4.7. *We have*

$$\mathcal{B}(k, d, \beta) = \mathcal{B}^{(\ell)}(k, d, \beta) + o_\ell(1).$$

Proof. The result easily follows from the weak convergence of $\pi^{(\ell)}$ toward $\pi_{k,d,\beta}^\star$. \square

We now proceed to prove Proposition 2.8. In order to do so, we need to state here a standard lemma about the local convergence of the random formula $\hat{\Phi}$, that we will prove in Section 6 (see Lemma 6.1).

Lemma 4.8. *For all $\ell \geq 0$ and $\forall T \in \widetilde{\mathcal{T}}_{2\ell+2}$, we have $\rho_\Phi(T) \sim \tilde{p}_{k,d,\beta}^{(2\ell+2)}(T)$.*

Proof of Proposition 2.8. By the previous lemma we have, for $\ell \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\mathcal{B}_{\Phi,\ell}] = \mathcal{B}^{(\ell)}(k, d, \beta).$$

The result then follows from Proposition 4.7. \square

5. MARGINAL ANALYSIS

We will exhibit a number of *deterministic* conditions (six in total) that entail the non-reconstruction property. Subsequently we are going to show that the random formulas $\hat{\Phi}$ and $\tilde{\Phi}$ enjoy these properties with high probability.

Then we will need some information on the local structure of a formula Φ . For a variable node $x \in V$ and $t \geq 0$ we let $\Delta^{(t)}\Phi(x)$ denote the t -neighborhood of x in Φ . For a tree $T \in \widetilde{\mathcal{T}}_{2\ell+2}$ (defined in Section 4.2) we define the empirical density of T by

$$\rho_\Phi(T) = \frac{1}{n} \sum_{i \in [n]} \mathbf{1}_{\Delta^{(2\ell+2)}\Phi(x_i) \cong T}.$$

We shall say that a regular k -SAT formula Φ satisfies property **Local Structure** if the following is true, for every ℓ large enough.

Local Structure: $\forall T \in \widetilde{\mathcal{T}}_{2\ell+2}$, $\rho_\Phi(T) \sim \tilde{p}_{k,d,\beta}^{(2\ell+2)}(T)$.

We shall also demand that Φ satisfies the **Cycles** property:

Cycles: There are $o(\sqrt{n})$ cycles of length at most $\sqrt{\ln n}$.

In order to proceed further, we need to introduce a few more notations, similar to the ones that we used in Section 3. Let Φ be fixed and V denote its set of vertices, F denote its set of edges and E denote its set of (undirected) edges. For $x \in V$ (resp. $a \in F$), we let

$$\begin{aligned}\partial_1 x &= \{a \in \partial x, b_{a,x} = -1\}, & \partial_{-1} x &= \{a \in \partial x, b_{a,x} = 1\}, & \partial_l x &= \{a \in \partial x, |\{y \in \partial a \setminus \{x\}, b_{a,y} = 1\}| = l\}, \\ \partial_1 a &= \{x \in \partial a, b_{a,x} = -1\}, & \partial_{-1} a &= \{x \in \partial a, b_{a,x} = 1\}.\end{aligned}$$

We also introduce $\partial_{1,l} x = \partial_1 x \cap \partial_l x$, $\partial_{-1,l} x = \partial_{-1} x \cap \partial_l x$, and for $a_0 \in \partial x$ fixed, $\partial_1(x, a_0) = \partial_1 x \setminus \{a_0\}$, $\partial_{-1}(x, a_0) = \partial_{-1} x \setminus \{a_0\}$, $\partial_{1,l}(x, a_0) = \partial_{1,l} x \setminus \{a_0\}$, $\partial_{-1,l}(x, a_0) = \partial_{-1,l} x \setminus \{a_0\}$.

We define the λ -core of Φ (in symbols: $\text{Core}_\lambda(\Phi)$) as the largest set W of variables such that all $x \in W$ satisfy the following conditions.

CR1: there are at least $k(1 - \frac{\lambda^{-1}}{100})$ clauses $a \in \partial_1 x$ such that $\partial_1 a = \{x\}$.

CR2: there are no more than $k \exp(-\beta) \left(1 + \frac{\lambda}{100}\right)$ clauses $a \in \partial x$ such that $|\partial_{-1} a| = k$.

CR3: for any $1 \leq l \leq k$ the number of $a \in \partial_{-1} x$ such that $|\partial_1 a| = l$ is bounded by $\lambda k^{l+3}/l!$.

CR4: there are no more than $\lambda k^{3/4}$ clauses $a \in \partial_1 x$ such that $|\partial_1 a| = 1$ but $\partial a \not\subseteq W$.

CR5: there are no more than $\lambda k^{3/4}$ clauses $a \in \partial_{-1} x$ such that $|\partial_{-1} a| < k$ and $|\partial_1 a \setminus W| \geq |\partial_1 a|/4$.

The λ -core is well-defined; for if W, W' satisfy the above conditions, then so does $W \cup W'$. Further, if $\lambda < \lambda'$, then $\text{Core}_\lambda(\Phi) \subset \text{Core}_{\lambda'}(\Phi)$. Also note the similarity with the trunk of trees defined in Section 3.2. We say that Φ satisfies the property **Core** if and only if

Core: $|\text{Core}_{1/2}(\Phi)| \geq (1 - 2^{-0.95})n$.

Our aim will be to identify a large set $V_{\text{good}} \subset V$ of vertices whose value under a typical assignment in the cluster is unlikely to be very far from the planted one. A first candidate for vertices whose marginal may go wrong are those which do not belong to the 1-core of Φ . Yet, we are not guaranteed that vertices in the core have marginals sufficiently close to $\mu^{(0)}$. For instance, if the marginals of most of the neighbors of a given vertex $x \in \text{Core}_1(\Phi)$ went astray, there would be no reason for x 's marginal not to go astray itself. However, we see that the vertices in the core whose marginals are not what we think they should be must clump together. We say that a set $S \subset V$ is λ -sticky if and only if for all $x \in S$, one of the following conditions holds true.

ST1: there are at least $\lambda k^{3/4}$ clauses $a \in \partial_1 x$ such that $\partial_1 a = \{x\}$ and $\partial_{-1} a \cap S \neq \emptyset$.

ST2: there are at least $\lambda k^{3/4}$ clauses $a \in \partial_{-1} x$ such that $|\partial_{-1} a| < k$ and $|\partial_1 a \cap S| \geq |\partial_1 a|/4$.

We say that Φ satisfies the property **Sticky** if and only if

Sticky: Φ has no $1/2$ sticky set of size between $2^{-0.95k}n$ and $2^{-k/20}n$.

Finally, say that a variable $x \in V$ is $(\varepsilon, 2\ell)$ -cold if the following is true. Let $T = \Delta^{2\ell} x$. Then T is a tree. Moreover, if we choose a boundary condition τ such that

- the values of variables y that do not belong to the core are chosen adversarially,
- the values of the other variables are chosen i.i.d. such that the probability of -1 equals $\exp(-2\beta)$,
- subsequently an adversary is allowed to change some of the -1 s to $+1$ s,

then with this boundary condition the BP marginal at the root of the tree is within ε of μ_T in total variation distance.

We say that Φ satisfies the property $(\varepsilon, 2\ell)$ -**Cold** iff

$(\varepsilon, 2\ell)$ -**Cold:** All but εn variables are $(\varepsilon, 2\ell)$ -cold.

We say that a formula Φ is (ε, ℓ) -tame iff the properties **Local Structure**, **Cycles**, **Core**, **Sticky** and $(\varepsilon, 2\ell)$ -**Cold** hold. Planted formulas are likely to be tame.

Proposition 5.1. *For any $\varepsilon > 0$, there is $\ell > 0$ such that w.h.p. $\hat{\Phi}$ is (ε, ℓ) -tame.*

Similarly, formulas from the planted replica model are likely to be tame as well.

Proposition 5.2. *For any $\varepsilon > 0$, there is $\ell > 0$ such that w.h.p. $\tilde{\Phi}$ is (ε, ℓ) -tame.*

We prove Propositions 5.1 and 5.2 in Section 6. We are going to show that (ε, ℓ) -tame formulas have the non-reconstruction property. In the rest of this section, we assume that Φ is (ε, ℓ) -tame. Let us briefly write $\llbracket \cdot \rrbracket = \llbracket \cdot \rrbracket_{\Phi, \beta}$ and $\langle \cdot \rangle = \langle \cdot \rangle_{\Phi, \beta}$.

We say that a set $T \subset \text{Core}_1(\Phi) \setminus S_1(\Phi)$ is σ -closed if for any $x \in T$ and all $a \in \partial x$ we have

$$\{y \in \partial a \cap \text{Core}_1(\Phi) \setminus S(\Phi) : \sigma(y) = -1\} \subset T.$$

Moreover, for a clause b we say $T \subset \text{Core}_1(\Phi) \setminus S(\Phi)$ is (σ, b) -closed if the above holds for all $x \in T$ and all $a \in \partial x \setminus b$.

Lemma 5.3. *Suppose that Φ is (ε, ℓ) -tame. Then for any σ such that $\mathbf{1} \cdot \sigma \geq (1 - 2^{-k/2})n$ and for any (σ, b) -closed set $T \subset \text{Core}_1(\Phi) \setminus S(\Phi)$ the following is true. Let $\tilde{\sigma}(x) = (-1)^{\mathbf{1}_{\{x \in T\}}} \sigma(x)$. Then*

$$E_\Phi(\tilde{\sigma}) \leq E_\Phi(\sigma) - k^{3/4}|T|. \quad (5.1)$$

Proof. Consider the following process:

- Let $\sigma_0 = \sigma$, $V_0 = T$ and $U_0 = \sigma^{-1}(-1) \setminus V_0$.
- While there is $i_t \in V_t$ such that $E_\Phi((-1)^{\mathbf{1}_{\{\cdot = i_t\}}} \sigma_t(\cdot)) \leq E_\Phi(\sigma_t) - k^{3/4}$, pick one such i_t uniformly at random and let $\sigma_{t+1}(\cdot) = (-1)^{\mathbf{1}_{\{\cdot = i_t\}}} \sigma_t(\cdot)$ and $V_{t+1} = V_t \setminus \{i_t\}$.

Clearly,

$$E_\Phi(\sigma_t) \leq E_\Phi(\sigma) - k^{3/4}t. \quad (5.2)$$

Let τ be the stopping time of this process and assume that $\tau < |T|$, or, in other words, that $V_\tau \neq \emptyset$. We claim that V_τ is a 1-sticky set. Indeed, because T is σ -closed for $i \in V_\tau$ we have

$$-k^{3/4} \leq E_\Phi((-1)^{\mathbf{1}_{\{\cdot = i\}}} \sigma_t(\cdot)) - E_\Phi(\sigma_\tau) \leq \mathbf{1}\{b \in \partial i\} - |\partial_{1,0} i| + |\{a \in \partial_{1,0} i, \partial_{-1} a \cap (V_\tau \cup U_0) \neq \emptyset\}|$$

$$+ |\partial_{-1,0} i| + |\cup_{1 \leq l \leq k} \{a \in \partial'_{-1,l}, \partial_1 a \in V_\tau \cup U_0\}|.$$

Because $i \in \text{Core}_1(\Phi)$ we have $|\partial_{1,0} i| \geq k^{7/8}$, $|\partial_{-1,0} i| \leq 3$, $|\{a \in \partial_{1,0}, \partial_{-1} a \cap U_0 \neq \emptyset\}| \leq k^{3/4}$ and $|\{a \in \partial_{1,0} i, |\partial_{-1} a \cap U_0| \geq |\partial_{-1} a|/4\}| \leq k^{3/4}$. Therefore, one of the following must hold.

- (a) $|\{a \in \partial_{1,0}, \partial_{-1} a \cap V_\tau \neq \emptyset\}| \geq k^{3/4}$,
- (b) $|\{a \in \partial_{1,0} i, |\partial_{-1} a \cap V_\tau| \geq |\partial_{-1} a|/4\}| \geq k^{3/4}$,

It follows that the set V_τ is 1-sticky. However, $\text{Core}_1(\Phi) \setminus S_1(\Phi)$ cannot contain a 1-sticky set of size $|V_\tau| \leq |T| \leq 2^{-k/20}$ as this would contradict the maximality of $S(\Phi)$. It follows that $\tau = |T|$, and therefore $\sigma_\tau = \bar{\sigma}$, from which (5.1) follows using (5.2). \square

Fact 5.4. *For any variable x the following is true. Let $\gamma(x, L)$ be the number of trees of order L rooted at x that are contained in the factor graph of Φ . Then $\gamma(x, L) \leq L(100dk)^L$.*

Write $T(x, \sigma)$ for the smallest σ -closed set that contains x . In other words, this is the -1 -component in $\text{Core}_1(\Phi) \setminus S_1(\Phi)$ that x belongs to. If $\sigma(x) = 1$ we let $T(x, \sigma) = \emptyset$.

Lemma 5.5. *If Φ is (ε, ℓ) -tame, then for all $x \in \text{Core}_1(\Phi) \setminus S_1(\Phi)$ we have*

$$\|\sigma(x)\| \geq 1 - \exp(-\beta k^{3/4}/2) \quad \text{and} \quad \|\mathbf{1}\{|T(x, \sigma)| > \ln \ln n\}\| \leq 1/\ln n.$$

Proof. Let $N = 2^{-k/2}n$. Because Φ is tame we have $\|\mathbf{1} \cdot \sigma < n - N\| \leq \exp(-\Omega(n))$. Therefore, $\|\mathbf{1}\{|T(x, \sigma)| > N\}\| \leq \exp(-\Omega(n))$. Hence, let $t < N$ and let θ be a tree of order t with root x that is contained in the factor graph of Φ and whose vertices lie in $\text{Core}_1(\Phi) \setminus S_1(\Phi)$. If σ is such that $T(x, \sigma) = \theta$, then Lemma 5.3 implies that $\bar{\sigma}(x) = (-1)^{\mathbf{1}\{x \in T(x, \sigma)\}} \sigma(x)$ satisfies $E_\Phi(\bar{\sigma}) \leq E_\Phi(\sigma) - k^{3/4}t$. Consequently,

$$\frac{\langle \mathbf{1}\{\sigma = \sigma\} \rangle}{\langle \mathbf{1}\{\sigma = \bar{\sigma}\} \rangle} \leq \exp(-\beta k^{3/4}t).$$

Hence, by Fact 5.4, the union bound and our assumptions on β and d ,

$$\frac{\langle \mathbf{1}\{|T(x, \sigma)| = t\} \rangle}{\langle \mathbf{1}\{\sigma(x) = 1\} \rangle} \leq t(100dk)^t \exp(-\beta k^{3/4}t) \leq \exp(-0.99\beta k^{3/4}t). \quad (5.3)$$

This bound readily implies the second assertion. To obtain the first assertion, we sum (5.3) over $1 \leq t \leq N$. \square

Fact 5.6. *Let $q \in (0, 1)$ and $L \geq 1$. Suppose that μ is a probability distribution on $\{\pm 1\}^L$ such that for any $i \in [L]$ and any $y_1, \dots, y_L \in \{\pm 1\}$ we have*

$$(1 - q)\mu(y_1, \dots, y_{i-1}, -1, y_{i+1}, \dots, y_L) \leq q\mu(y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_L).$$

Furthermore, let ν be the distribution on $\{\pm 1\}^L$ such that for all $y_1, \dots, y_L \in \{\pm 1\}$ we have

$$(1 - q)\nu(y_1, \dots, y_{i-1}, -1, y_{i+1}, \dots, y_L) = q\nu(y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_L).$$

Moreover, let $B \subset \{\pm 1\}^L$ be a set such that for all $b \in B$ and all $b' \geq b$ we have $b' \in B$. Then $\mu(B) \geq \nu(B)$.

Lemma 5.7. *Let r be a variable for which the following conditions hold.*

- (1) r is (ε, ℓ) -cold.
- (2) r has distance at least $\ln^{1/3} n$ from any cycle of length at most $\sqrt{\ln n}$.

Let Γ_r be the event that σ is a good boundary condition for r . Then $\langle \mathbf{1}\{\sigma \notin \Gamma_r\} \rangle \leq 2\varepsilon$.

Proof. Let X be the set of boundary variables. Moreover, let \mathcal{A} be the event that $\max_{x \in X} |T(x, \sigma)| \leq \ln \ln n$ and that $\sigma \cdot \mathbf{1} \geq (1 - 2^{-k/2})n$. Because ℓ is bounded, Lemma 5.5 and the union bound imply that $\|\mathbf{1}\{\sigma \in \mathcal{A}\}\| \sim 1$. Furthermore, if \mathcal{A} occurs, then our assumption ensures that the subgraph of the factor graph induced on $Y = \partial^\ell r \cup \bigcup_{x \in X} T(x, \sigma)$ is acyclic.

Now, fix a variable $x \in X$ and $\sigma \in \mathcal{A}$ such that $\sigma(x) = -1$. Let a be the clause that is adjacent to x on its shortest path to r and let $T(x, a, \sigma)$ be the smallest (σ, a) -closed set that contains x . Further, define $\bar{\sigma}(y) = (-1)^{\mathbf{1}\{y \in T(x, \sigma)\}}$. Then Lemma 5.3 shows that $E_\Phi(\bar{\sigma}) \leq k^{3/4}|T(x, a, \sigma)|$. Moreover, because the subgraph induced on Y is acyclic we have $\bar{\sigma}(x') = \sigma(x')$ for all $x' \in X \setminus \{x\}$. Consequently, by Fact 5.4 and the union bound,

$$\frac{\langle \mathbf{1}\{\sigma(x) = -1\} \prod_{y \in X \setminus \{x\}} \mathbf{1}\{\sigma(y) = \sigma(y)\} \mathbf{1}\{\sigma \in \mathcal{A}\} \rangle}{\langle \mathbf{1}\{\sigma(x) = 1\} \prod_{y \in X \setminus \{x\}} \mathbf{1}\{\sigma(y) = \sigma(y)\} \rangle} \leq \sum_{t \leq \ln \ln n} t(100dk)^t \exp(-\beta k^{3/4}t) \leq \exp(-\beta k^{3/4}/2). \quad (5.4)$$

Since $\llbracket \mathbf{1}\{\sigma \in \mathcal{A}\} \rrbracket \sim 1$ and because for all $\tau : X \rightarrow \{\pm 1\}$ we have

$$\left\langle \prod_{y \in X \setminus \{x\}} \mathbf{1}\{\sigma(y) = \tau(y)\} \right\rangle \geq \exp(-dk\beta|X|) = \Omega(1),$$

(5.4) implies that for any τ ,

$$\frac{\langle \mathbf{1}\{\sigma(x) = -1\} \prod_{y \in X \setminus \{x\}} \mathbf{1}\{\sigma(y) = \tau(y)\} \rangle}{\langle \mathbf{1}\{\sigma(x) = 1\} \prod_{y \in X \setminus \{x\}} \mathbf{1}\{\sigma(y) = \tau(y)\} \rangle} \leq \exp(-\beta k^{3/4}/3).$$

Thus, the assertion follows from Fact 5.6. \square

Finally, Propositions 2.5 and 2.6 follow from Propositions 5.1 and 5.2 and Lemma 5.7.

6. TYPICAL PROPERTIES OF THE RANDOM FORMULA

In this section we prove Proposition 5.1 and Proposition 5.2. Let $\mathcal{E}_{n,k,d}$ denote the set of regular k -SAT formulas. For $v \in V \cup F$ and $\ell \geq 0$, we let $\partial^\ell v$ (resp. $\Delta^\ell v$) denote the set of vertices at distance exactly ℓ (resp. less than ℓ) from v .

6.1. Proof of Proposition 5.1 and Proposition 5.2. We first deal with the easiest condition **Local Structure**.

Lemma 6.1. *W.h.p. $\hat{\Phi}$ satisfies Local Structure.*

Proof. Let $x \in V$ and $T \in \widetilde{\mathcal{T}}_{2(\ell+1)}$ be fixed. Let $X_x(T)$ be the number of formulas Φ such that $\Delta^{2(\ell+1)}\Phi(x) = T$. It is straightforward to compute that there are precisely

$$\tilde{p}_{k,d,\beta}^{2(\ell+1)}(T) \frac{(nd/2)!^2}{(nd/2 - \epsilon_+)!(nd/2 - \epsilon_-)!} (1 + o_n(1))$$

ways to construct a tree of depth $2(\ell+1)$ around x , where ϵ_+ (resp. ϵ_-) is the number of positive (resp. negative) literals that appear in $T \setminus \partial T$. Once this has been done, it remains to connect the $(dn/2 - \epsilon_+)$ positive literals clones (resp. $(dn/2 - \epsilon_-)$ negative literals clones) together. This yields

$$\frac{X_x(T)}{|\mathcal{E}_{n,k,d}|} = \tilde{p}_{k,d,\beta}^{(\ell+1)}(T) \frac{(nd/2)!^2}{(nd/2 - \epsilon_+)!(nd/2 - \epsilon_-)!} \frac{(nd/2 - \epsilon_+)!(nd/2 - \epsilon_-)!}{(nd/2)!^2} (1 + o_n(1)) = \tilde{p}_{k,d,\beta}^{(\ell+1)}(T).$$

Consequently, we have

$$\mathbb{E}[\rho_\Phi(T)] = \frac{X_x(T)}{|\mathcal{E}_{n,k,d}|} = \tilde{p}_{k,d,\beta}^{2(\ell+1)}(T).$$

Moreover by standard concentration arguments $\rho_\Phi(T)$ is concentrated around its mean and we have w.h.p.

$$\rho_\Phi(T) \sim \tilde{p}_{k,d,\beta}^{2(\ell+1)}(T).$$

This holds for any T in the finite set $\widetilde{\mathcal{T}}_{2(\ell+1)}$, ending the proof of the lemma. \square

In particular, this entails the following.

Corollary 6.2. *W.h.p. $\tilde{\Phi}$ satisfies Local Structure.*

The following is a standard result.

Fact 6.3. *W.h.p. $\hat{\Phi}$ and $\tilde{\Phi}$ satisfy the property Cycles.*

We will prove the following in Section 6.3.

Proposition 6.4. *W.h.p. $\hat{\Phi}$ and $\tilde{\Phi}$ satisfy Core and Sticky.*

The remaining of this section is devoted to a proof of the two following lemmas.

Lemma 6.5. *For all $\epsilon > 0$, there is $\ell > 0$ such that w.h.p. $\hat{\Phi}$ is (ϵ, ℓ) -cold.*

Lemma 6.6. *For all $\epsilon > 0$, there is $\ell > 0$ such that w.h.p. $\tilde{\Phi}$ is (ϵ, ℓ) -cold*

Proof of Proposition 5.1 and Proposition 5.2. The propositions immediately follow from the above lemmas. \square

Let $\alpha \geq 0$ and $(z_1, \dots, z_\alpha) \in V^\alpha$ be fixed. Let $\Delta = \{y \in V, \exists l \in [\alpha], y \in \partial^{(2\ell)} \widehat{\Phi}(z_l)\}$ and for $y \in \Delta$ let \mathcal{C}_y be the event that $y \in \text{Core}_1(\widehat{\Phi}) \setminus S_1(\widehat{\Phi})$. Moreover, let \mathcal{D} be the event that $z_1, z_2, \dots, z_\alpha$ are at distance strictly greater than 5ℓ one from the other in $\widehat{\Phi}$, and that their 5ℓ neighborhoods are tree-like. For $y \in \Delta$, let also \mathcal{F}_y denote the σ -algebra generated by the function $(\Phi, z_1, \dots, z_\alpha) \mapsto (\Delta^{(2\ell)} \Phi(z_1) \cup \dots \cup \Delta^{(2\ell)} \Phi(z_\alpha))$.

Lemma 6.7. *For $y \in \Delta$, we have*

$$\mathbb{P}[\neg \mathcal{C}_y | \mathcal{D}, \mathcal{F}_y] \leq 2^{-0.95k}.$$

Proof. Let $a_y \in \partial y$ be such that $a_y \in \cup_{l \in [\alpha]} \Delta^{(2\ell)} \widehat{\Phi}(z_l)$. Let $\widehat{\Phi}'$ be obtained from $\widehat{\Phi}$ by the following operations.

- Select $x \in \widehat{\Phi}$ and $a_x \in \partial x$ uniformly at random.
- Replace the pair of edges $\{(y, a_y), (x, a_x)\}$ by the pair of edges $\{(y, a_x), (x, a_y)\}$.

Let \mathcal{E} be the event that $\widehat{\Phi}'$ satisfies \mathcal{D} . We observe that $\mathbb{P}[\mathcal{D}] = 1 - o_n(1)$ and $\mathbb{P}[\mathcal{E}] = 1 - o_n(1)$. Conditioned on \mathcal{D}, \mathcal{E} and \mathcal{F}_y , $\widehat{\Phi}$ and $\widehat{\Phi}'$ are identically distributed. Moreover, we have

$$\text{Core}_{1/2}(\widehat{\Phi}) \setminus S_{1/2}(\widehat{\Phi}) \subset \text{Core}_1(\widehat{\Phi}') \setminus S(\widehat{\Phi}').$$

It follows that

$$\begin{aligned} \mathbb{P}[\neg \mathcal{C}_y | \mathcal{D}, \mathcal{F}_y] &= \mathbb{P}[\neg \mathcal{C}_y | \mathcal{D}, \mathcal{E}, \mathcal{F}_y] + o_n(1) \\ &\leq \mathbb{P}[x \notin \text{Core}_{1/2}(\widehat{\Phi}) \setminus S_{1/2}(\widehat{\Phi}) | \mathcal{D}, \mathcal{E}, \mathcal{F}_y] + o_n(1) \\ &\leq \mathbb{P}[x \notin \text{Core}_{1/2}(\widehat{\Phi}) \setminus S_{1/2}(\widehat{\Phi}) | \mathcal{F}_y] + o_n(1). \end{aligned} \quad (6.1)$$

For a fixed σ -algebra \mathcal{F} generated by $(\Phi, z_1, \dots, z_\alpha) \mapsto (\Delta^{(2\ell)} \Phi(z_1) \cup \dots \cup \Delta^{(2\ell)} \Phi(z_\alpha))$, let \mathcal{H} denote the event that there is $\widehat{\Phi}''$ isomorphic to $\widehat{\Phi}$ such that $\mathcal{F}_y = \mathcal{F}$. Then, because x is a random element of V , we have

$$\begin{aligned} \mathbb{P}[x \notin \text{Core}_{1/2}(\widehat{\Phi}) \setminus S_{1/2}(\widehat{\Phi}) | \mathcal{F}_y = \mathcal{F}] &= \mathbb{P}[x \notin \text{Core}_{1/2}(\widehat{\Phi}) \setminus S_{1/2}(\widehat{\Phi}) | \mathcal{H}] \\ &= \mathbb{P}[x \notin \text{Core}_{1/2}(\widehat{\Phi}) \setminus S_{1/2}(\widehat{\Phi})] + o_n(1), \end{aligned} \quad (6.2)$$

where the last line used that $\mathbb{P}[\mathcal{H}] = 1 - o_n(1)$. Finally, Proposition 6.4 implies that

$$\mathbb{P}[x \notin \text{Core}_{1/2}(\widehat{\Phi}) \setminus S_{1/2}(\widehat{\Phi})] \leq 2^{1-0.96k} + o_n(1). \quad (6.3)$$

Combining (6.1), (6.2) and (6.3) concludes the proof of the lemma. \square

Proof of Lemma 6.5. For $\varepsilon > 0$ fixed, let $\mathbf{Y} = |\{x \in V, \Delta^{(2\ell)} \widehat{\Phi}(x) \text{ is not } (\varepsilon, 2\ell)\text{-cold in } \widehat{\Phi}\}|$, and let $\alpha(n)$ be a slowly diverging function. We are going to show that there is a sequence $y_\ell = o_\ell(1)$ such that

$$\mathbb{E}[\mathbf{Y}(\mathbf{Y} - 1) \dots (\mathbf{Y} - \alpha + 1)] \leq (y_\ell n)^\alpha. \quad (6.4)$$

This bound implies the assertion; indeed,

$$\begin{aligned} \mathbb{P}[\mathbf{Y} > 3y_\ell n] &\leq \mathbb{P}[\mathbf{Y}(\mathbf{Y} - 1) \dots (\mathbf{Y} - \alpha + 1) > (2y_\ell n)^\alpha] \\ &\leq \frac{\mathbb{E}[\mathbf{Y}(\mathbf{Y} - 1) \dots (\mathbf{Y} - \alpha + 1)]}{(2y_\ell n)^\alpha} \leq 2^{-\alpha}. \end{aligned}$$

To prove (6.4), we observe that $\mathbf{Y}(\mathbf{Y} - 1) \dots (\mathbf{Y} - \alpha + 1)$ is just the number of orderer α -tuples of variables such that $\Delta^{(2\ell)} \Phi(x)$ is not $(\varepsilon, 2\ell)$ -cold. Hence, by symmetry and linearity of expectation,

$$\mathbb{E}[\mathbf{Y}(\mathbf{Y} - 1) \dots (\mathbf{Y} - \alpha + 1)] \leq n^\alpha \mathbb{P}[\mathbf{T}_1, \dots, \mathbf{T}_\alpha \text{ are not } (\varepsilon, 2\ell)\text{-cold}],$$

where $\mathbf{T}_1, \dots, \mathbf{T}_\alpha$ are the 2ℓ -neighborhoods chosen of α random vertices $\mathbf{x}_1, \dots, \mathbf{x}_\alpha$ of V . Let \mathcal{D} be the event that $\mathbf{x}_1, \dots, \mathbf{x}_\alpha$ are at distance greater than 5ℓ from each others and have tree-like 5ℓ neighborhoods, and let $\Delta = \partial \mathbf{T}_1 \cup \dots \cup \partial \mathbf{T}_\alpha$. Then Lemma 6.7 implies that for $j \in \Delta$

$$\mathbb{P}[\neg \mathcal{C}_j | \mathcal{D}, \mathcal{F}_j] \leq 2^{-0.95k}.$$

In particular, using Lemma 6.1 we can apply the result of Proposition 3.2 to obtain that, for $1 \leq i \leq \alpha$,

$$\mathbb{P}[\mathbf{T}_i \text{ is not } (\varepsilon, 2\ell)\text{-cold} | \mathcal{D}, \mathbf{T}_1, \dots, \mathbf{T}_{i-1}, \mathbf{T}_{i+1}, \mathbf{T}_\alpha] \leq \ell^{-1}. \quad (6.5)$$

We have

$$\begin{aligned} \mathbb{P}[\mathbf{T}_1, \dots, \mathbf{T}_\alpha \text{ are not } (\varepsilon, 2\ell)\text{-cold}] &\leq \mathbb{P}[\mathbf{T}_1 \text{ is not } (\varepsilon, 2\ell)\text{-cold} | \mathcal{D}] \mathbb{P}[\mathbf{T}_2 \text{ is not } (\varepsilon, 2\ell)\text{-cold} | \mathbf{T}_1 \text{ is not } (\varepsilon, 2\ell)\text{-cold}, \mathcal{D}] \dots \\ &\quad \mathbb{P}[\mathbf{T}_\alpha \text{ is not } (\varepsilon, 2\ell)\text{-cold} | \mathbf{T}_1, \dots, \mathbf{T}_{\alpha-1} \text{ are not } (\varepsilon, 2\ell)\text{-cold}, \mathcal{D}]. \end{aligned}$$

Using (6.5) this yields

$$\mathbb{P}[T_1, \dots, T_\alpha \text{ are not } (\varepsilon, 2\ell)\text{-cold} | \mathcal{D}] \leq (o_\ell(1))^\alpha.$$

Along with the observation that $\mathbb{P}[\mathcal{D}] = 1 - o_n(1)$, this concludes the proof of the proposition. \square

In order to prove Lemma 6.6, we need to extend Lemma 6.7 to the planted replica model. This will require a few more auxiliary results. We say that a tree $T \in \mathcal{T}_\ell$ is 2ℓ -pure if and only if

$$v_T^{(2\ell)}(1) \geq 1 - \exp(-100\beta).$$

Let $\mathcal{T}_{2\ell}^+ \subset \mathcal{T}_{2\ell}$ denote the set of pure trees. Let, as before, $\alpha \geq 0$ and $(z_1, \dots, z_\alpha) \in V^\alpha$ be fixed, as well as a formula and an assignment $(\Phi, \sigma) \in \mathcal{E}_{n,k,d} \times \{-1, 1\}^n$. Let $\Delta = \{y \in V, \exists l \in [\alpha], y \in \partial^{(2\ell)} \Phi(z_l)\}$ and for $y \in \Delta$, let a_y be the unique clause in $\partial y \cap \bigcup_{l=1}^\alpha \Delta^{(2\ell)} \Phi(z_l)$ and let $l_y \in [k]$ (resp. $l'_y \in [d]$) be such that y appears in l_y -th position in a_y (resp. a_y appears in l'_y position in y). For $y \in \Delta$, let $\Delta^{(2\ell)} \tilde{\Phi}(y \rightarrow a_y)$ denote the 2ℓ neighborhood of y in the formula where the edge between y and a_y has been removed and let

- \mathcal{A}_y be the event that $\Delta^{(2\ell)} \tilde{\Phi}(y \rightarrow a_y)$ is tree-like and is 2ℓ -pure in $\tilde{\Phi}$,
- \mathcal{B}_y be the event that $\tilde{\sigma}(y) = 1$,
- \mathcal{C}_y be the event that $y \in \text{Core}_{1/2}(\tilde{\Phi}) \setminus S_{1/2}(\tilde{\Phi})$.

Moreover, let \mathcal{D} be the event that z_1, \dots, z_α are at distance greater than 5ℓ in $\tilde{\Phi}$. For $y \in \Delta$, let also \mathcal{G}_y denote the sigma algebra induced by the functions

$$(\Phi, z_1, \dots, z_\alpha, y) \mapsto \left(X = \Delta^{(2\ell)} \Phi(z_1) \cup \dots \cup \Delta^{(2\ell)} \Phi(z_\alpha) \setminus \Delta^{(2\ell)} \Phi(y \rightarrow a_y), \sigma|_X \right).$$

With these notations in mind, we will prove the following.

Lemma 6.8. *For $y \in \Delta$, we have*

$$\mathbb{P}[\neg \mathcal{A}_y | \mathcal{D}, \mathcal{G}_y] \leq 2^{-0.95k}.$$

Proof. This lemma follows from Lemma 3.3 and Lemma 3.6, using in addition the fact that $\mathbb{P}[\mathcal{D}] = 1 - o_n(1)$. \square

Lemma 6.9. *For $y \in \Delta$, we have*

$$\mathbb{P}[\neg \mathcal{B}_y | \mathcal{A}_y, \mathcal{D}, \mathcal{G}_y] \leq 4^{-k}.$$

Proof. Let $\widetilde{\mathcal{T}}_{2\ell+1}$ denote the set of $2\ell + 1$ neighborhoods of the first children of the root of the Galton-Watson process $\text{GW}'(k, d, \beta, 4\ell)$ (defined in Section 3). For a probability distribution μ over $\{-1, 1\}^k$ and $1 \leq l \leq k$, let $\mu[l]$ denote the l -th marginal of μ . Finally, recall that for $a \in F$, $\mu_a^{(2\ell+1)}$ was defined in Section 2.3.

Recalling the definition of the replica planted model, we have

$$\mathbb{P}[\neg \mathcal{B}_y | \mathcal{A}_y, \mathcal{D}, \mathcal{G}_y] = \sum_{\hat{T} \in \widetilde{\mathcal{T}}_\ell} \mathbb{P}[\Delta^{(2\ell+1)} \tilde{\Phi}(a_y) = \hat{T} | \mathcal{A}_y, \mathcal{D}, \mathcal{G}_y] \hat{\mu}_a^{(2\ell+1)}[l_y](-1).$$

For $\hat{T} \in \widetilde{\mathcal{T}}_{2\ell+1}$ and $1 \leq l \leq k$, let $\hat{T}[l]$ denote the subtree of size 2ℓ rooted at the l -th variable node adjacent to the root of \hat{T} . For $T \in \mathcal{T}_{2\ell}^+$, let $\widehat{\mathcal{T}}_{2\ell+1}(T, l) \subset \widetilde{\mathcal{T}}_{2\ell+1}$ denote the set of trees compatible with T on l -th position:

$$\widehat{\mathcal{T}}_{2\ell+1}(T, l) = \left\{ \hat{T} \in \widetilde{\mathcal{T}}_{2\ell+1}, \hat{T}[l] = T \right\}.$$

Then we immediately deduce from the previous equation that

$$\mathbb{P}[\neg \mathcal{B}_y | \mathcal{A}_y, \mathcal{F}_y] \leq \sup_{T \in \mathcal{T}_{2\ell}^+} \sup_{\hat{T} \in \widehat{\mathcal{T}}_{2\ell+1}(T, l_y)} \hat{\mu}_{\hat{T}}^{(2\ell+1)}[l](-1)(1 + o_n(1))$$

Using the definition of ℓ -pure trees, and the observation that marginals and messages cannot differ by a factor of more than $\exp(\beta)$, for any $T \in \mathcal{T}_{2\ell}^+$ and $\hat{T} \in \widehat{\mathcal{T}}_\ell(T, l)$ we have $\hat{\mu}_{\hat{T}}^{(2\ell+1)}[l](-1) \leq \exp(-50\beta) \leq 4^{-k}$, ending the proof of the lemma. \square

Lemma 6.10. *For $y \in \Delta$, we have*

$$\mathbb{P}[\neg \mathcal{C}_y | \mathcal{A}_y, \mathcal{B}_y, \mathcal{D}, \mathcal{G}_y] \leq 2^{-0.95k}.$$

Proof. Let $\tilde{\Phi}'$ be obtained from $\tilde{\Phi}$ by the following operation.

- Select $x \in \tilde{\Phi}$ such that $\tilde{\sigma}_x = \tilde{\sigma}_y$ and (denoting by a_x the l'_y -th clause adjacent to x) $\Delta^{(4\ell)} \tilde{\Phi}(x, a_x) = \Delta^{(4\ell)} \tilde{\Phi}(y, a_y)$ at random.
- Replace the pair of edges $\{(y, a_y), (x, a_x)\}$ by the pair of edges $\{(y, a_x), (x, a_y)\}$.

Let \mathcal{E} be the event that $\tilde{\Phi}'$ satisfies \mathcal{D} . We observe that $\mathbb{P}[\mathcal{D}] = 1 - o_n(1)$ and $\mathbb{P}[\mathcal{E}] = 1 - o_n(1)$. Conditioned on \mathcal{D}, \mathcal{E} and \mathcal{G}_y , $\tilde{\Phi}$ and $\tilde{\Phi}'$ are identically distributed. Moreover, we have

$$\text{Core}_{1/2}(\tilde{\Phi}) \setminus S_{1/2}(\tilde{\Phi}) \subset \text{Core}_1(\tilde{\Phi}') \setminus S_1(\tilde{\Phi}').$$

It follows that

$$\begin{aligned} \mathbb{P}[\neg \mathcal{C}_y | \mathcal{A}_y, \mathcal{B}_y, \mathcal{D}, \mathcal{E}, \mathcal{G}_y] &= \mathbb{P}[\neg \mathcal{C}_y | \mathcal{A}_y, \mathcal{B}_y, \mathcal{D}, \mathcal{E}, \mathcal{G}_y] (1 + o_n(1)) \\ &\leq \sum_{T' \in \mathcal{T}_{4\ell}^+} \mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{A}_y, \mathcal{B}_y, \mathcal{D}, \mathcal{E}, \mathcal{G}_y] \\ &\quad \mathbb{P}[x \notin \text{Core}_{1/2}(\tilde{\Phi}) \setminus S_{1/2}(\tilde{\Phi}) | \mathcal{B}_x, \mathcal{D}, \mathcal{E}, \mathcal{G}_y, \Delta^{(4\ell)} \tilde{\Phi}(x, a_x) = T']. \end{aligned} \quad (6.6)$$

We define, for $T' \in \mathcal{T}_{4\ell}^+$, $T(T') \in \widehat{\mathcal{T}}_{4\ell+1}$ by $T[l'_y] = \Delta^{(4\ell)} \Phi(a_y \rightarrow y)$ and $T[l] = T'[l]$ for $l' \neq l'_y$ (where $T[l]$ denotes the l -th subtree pending on T 's root). It follows from the same argument as previously for $T' \in \mathcal{T}_{4\ell}^+$ we have $\mu_{T(T')}^{(2\ell+1)}[l_y](1) \geq 1/2$. Therefore

$$\begin{aligned} \mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{A}_y, \mathcal{B}_y, \mathcal{D}, \mathcal{E}, \mathcal{G}_y] &= \frac{\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{A}_y, \mathcal{G}_y] \mu_{T(T')}^{(2\ell+1)}[l_y](1)}{\sum_{T'' \in \mathcal{T}_{4\ell}^+} \mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T'' | \mathcal{A}_y, \mathcal{G}_y] \mu_{T(T'')}^{(2\ell+1)}[l_y](1)} (1 + o_n(1)) \\ &\leq \frac{\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{A}_y, \mathcal{G}_y] \mu_{T(T')}^{(2\ell+1)}[l_y](1)}{\sum_{T'' \in \mathcal{T}_{4\ell}^+} \mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T'' | \mathcal{A}_y, \mathcal{G}_y] 1/2} \\ &\leq 2\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{A}_y, \mathcal{G}_y], \end{aligned}$$

Moreover

$$\begin{aligned} \mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{A}_y, \mathcal{G}_y] &\leq \\ &\leq \mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{A}_y]^{-1} \mathbb{P}[\mathcal{G}_y]^{-1} \\ &\leq 2\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T'] = 2\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T']. \end{aligned}$$

where we used Lemma 6.10 to obtain the second inequality. Using Baye's theorem once more, we have

$$\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T'] = \frac{\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{B}_y] \mathbb{P}[\mathcal{B}_y]}{\mathbb{P}[\mathcal{B}_y | \Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T']} \leq 2\mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{B}_y].$$

In order to deduce the last inequality, we used that by an argument similar to Lemma 6.9, $\mathbb{P}[\mathcal{B}_y | \Delta^{(4\ell)} \tilde{\Phi}(y, a_y)] \geq 1/2$. It follows by replacing in (6.6) that

$$\begin{aligned} \mathbb{P}[\neg \mathcal{C}_y | \mathcal{A}_y, \mathcal{B}_y, \mathcal{D}, \mathcal{E}, \mathcal{G}_y] &\leq 6 \sum_{T' \in \mathcal{T}_{4\ell}^+} \mathbb{P}[\Delta^{(4\ell)} \tilde{\Phi}(y, a_y) = T' | \mathcal{B}_y] \mathbb{P}[x \notin \text{Core}_{1/2}(\tilde{\Phi}) \setminus S_{1/2}(\tilde{\Phi}) | \mathcal{B}_x, \mathcal{D}, \mathcal{E}, \Delta^{(4\ell)} \tilde{\Phi}(x, a_x) = T'] \\ &\leq 6\mathbb{P}[x \notin \text{Core}_{1/2}(\tilde{\Phi}) \setminus S_{1/2}(\tilde{\Phi}) | \mathcal{B}_x, \mathcal{D}, \mathcal{E}] \\ &\leq 6\mathbb{P}[x \notin \text{Core}_{1/2}(\tilde{\Phi}) \setminus S_{1/2}(\tilde{\Phi})] \mathbb{P}[\mathcal{B}_x]^{-1} \mathbb{P}[\mathcal{D}]^{-1} \mathbb{P}[\mathcal{E}]^{-1} \\ &\leq 8\mathbb{P}[x \notin \text{Core}_{1/2}(\tilde{\Phi}) \setminus S_{1/2}(\tilde{\Phi})]. \end{aligned}$$

We used Lemma 6.9 to deduce the last inequality. Along with Proposition 6.4, this ends the proof of the lemma. \square

Proof of Proposition 5.2. We take a path similar to the proof of Proposition 5.1. Let $\varepsilon > 0$ be fixed. Let

$$Y = |\{x \in V, \Delta^{(2\ell)} \tilde{\Phi}(x) \text{ is not } (\varepsilon, 2\ell)\text{-cold in } \tilde{\Phi}\}|,$$

and let $\alpha(n)$ be a slowly diverging function. We are going to show that there is a sequence $y_\ell = o_\ell(1)$ such that

$$\mathbb{E}[Y(Y-1) \dots (Y-\alpha+1)] \leq (y_\ell n)^\alpha. \quad (6.7)$$

This bound implies the assertion as previously. As before, we observe that

$$\mathbb{E}[Y(Y-1)\dots(Y-\alpha+1)] \leq n^\alpha \mathbb{P}[T_1, \dots, T_\alpha \text{ are not } (\varepsilon, 2\ell)\text{-cold}],$$

where T_1, \dots, T_α are 2ℓ -neighborhoods of α random vertices $\mathbf{x}_1, \dots, \mathbf{x}_\alpha$ of V . Let \mathcal{D} be the event that $\mathbf{x}_1, \dots, \mathbf{x}_\alpha$ are at distance greater than 5ℓ from each others and have tree-like 5ℓ neighborhoods, and let $\Delta = \partial T_1 \cup \dots \cup \partial T_\alpha$. By combining Lemma 6.8, Lemma 6.9, Lemma 6.10 we obtain that for $y \in \Delta$

$$\mathbb{P}[\neg \mathcal{C}_y | \mathcal{D}, \mathcal{G}_y] \leq 2^{-0.94k}.$$

In particular, using Corollary 6.2 we can apply the result of Proposition 3.2 to obtain that, for $1 \leq i \leq \alpha$,

$$\mathbb{P}[T_i \text{ is not } (\varepsilon, 2\ell)\text{-cold} | \mathcal{D}, T_1, \dots, T_{i-1}, T_{i+1}, T_\alpha] \leq \ell^{-1}. \quad (6.8)$$

We have

$$\begin{aligned} \mathbb{P}[T_1, \dots, T_\alpha \text{ are not } (\varepsilon, 2\ell)\text{-cold} | \mathcal{D}] &\leq \mathbb{P}[T_1 \text{ is not } (\varepsilon, 2\ell)\text{-cold} | \mathcal{D}] \mathbb{P}[T_2 \text{ is not } (\varepsilon, 2\ell)\text{-cold} | T_1 \text{ is not } (\varepsilon, 2\ell)\text{-cold}, \mathcal{D}] \dots \\ &\quad \mathbb{P}[T_\alpha \text{ is not } (\varepsilon, 2\ell)\text{-cold} | T_1, \dots, T_{\alpha-1} \text{ is not } (\varepsilon, 2\ell)\text{-cold}, \mathcal{D}]. \end{aligned}$$

Using (6.8) yields

$$\mathbb{P}[T_1, \dots, T_\alpha \text{ are not } (\varepsilon, 2\ell)\text{-cold} | \mathcal{D}] \leq (o_\ell(1))^\alpha.$$

Along with the observation that $\mathbb{P}[\mathcal{D}] = 1 - o_n(1)$, this concludes the proof of the proposition. \square

6.2. Proof of Proposition 6.4. In order to prove Proposition 6.4, we will identify a set of simpler events that will imply the proposition. We will first need to control the number of vertices with unusual 2-neighborhood. To this end, we let for a formula Φ , U_0 be the set of variables such that $|\{a \in \partial_{1,0}x\}| < 2k^{7/8}$, $|\partial_{-1,0}x| \geq 2$ or such that there exists $1 \leq l \leq k$ such that $|\{\partial_{-1,l}x\}| \geq 0.01k^{l+3}/l!$. Our first condition will ensure that U_0 is not too large:

$$|U_0| \leq 2^{-0.98k}n. \quad (\mathcal{C}0)$$

We now turn to expansion properties of Φ . We define, for a set $T \subset V$ the sets

$$\begin{aligned} F_0(T) &= \{a \in F, |\partial_1 a| = 1, \partial_1 a \subset T, |\partial_{-1}(a) \cap T| \geq 1\}, \\ F_l(T) &= \{a \in F, \partial_{-1}a \cap T \neq \emptyset, |\partial_1 a| = l, |\partial_1 a \cap T| \geq l/4\} \quad (\text{for } 1 \leq l \leq k). \end{aligned}$$

The following conditions encompass bounds on the sizes of the sets $F_i(T)$ when T has moderate size.

$$\text{There is no set } T \subset V \text{ of size } |T| \in [2^{-0.97k}n, 2^{-k/20}n] \text{ and such that } |F_0(T)| \geq k^{3/4}|T|/100. \quad (\mathcal{C}1)$$

$$\text{For each } 1 \leq l \leq k, \text{ there is no set } T \subset V \text{ of size } |T| \in [2^{-0.97k}n, 2^{-k/20}n] \quad (\mathcal{C}2)$$

$$\text{and such that } |F_l(T)| \geq k^{3/4}|T|/(100l^2).$$

Lemma 6.11. *Assume that Φ satisfies $(\mathcal{C}0)$ -($\mathcal{C}2$). Then it satisfies **Core** and **Sticky**.*

Proof. Let Φ be such that it satisfies $(\mathcal{C}0)$ -($\mathcal{C}2$). We first prove that Φ does not admit a 1/2-sticky set S with $|S| \in [2^{-0.97k}n, 2^{-k/20}n]$. Indeed, let $S \subset V$ be a 1/2-sticky set for Φ and let

$$\begin{aligned} S_0 &= \{x \in S, |\{a \in \partial_{1,0}x, \partial(a, x) \cap S \neq \emptyset\}| \geq k^{3/4}/2\}, \\ S_l &= \{x \in S, |\{a \in \partial_{-1,l}x, |\partial_1(a, x)| = l, |\partial_1(a, x) \cap S| \geq l/4\}| \geq k^{3/4}/2\} \quad (\text{for } 1 \leq l \leq k-1). \end{aligned}$$

We first observe that

$$|F_0(S)| \geq k^{3/4}|S_0|/2, \quad \text{and that for } 1 \leq l \leq k-1 \quad |F_l(S)| \geq k^{3/4}|S_l|/(2l). \quad (6.9)$$

Because S is 1/2-sticky, we have $S \subset S_0 \cup \bigcup_{l=1}^{k-1} S_l$ and therefore either $|S_1| \geq |S|/100$ or there is $1 \leq l \leq k-1$ such that $|S_l| \geq |S|/(100l^2)$. In either case, it follows from (6.9) and $(\mathcal{C}1)$ -($\mathcal{C}2$) that $|S_l| \notin [2^{-0.98k}n, 2^{-k/20}n]$. Using that (for $0 \leq l \leq k-1$) $|S_l| \leq |S| \leq k|S_l|$ shows that S has size outside the range $[2^{-0.97k}n, 2^{-k/20}n]$.

We now turn to the study of the 1/2-core of Φ . Given Φ , we consider the following *whitening* process. Let $U = U_0$ initially. While there is a variable $x \notin U$ such that one of the following conditions occurs, add x to U .

- (a) $|\{a \in \partial_{1,0}x, |\partial_{-1,0}a \cap U| \geq 1\}| > k^{3/4}/2$.
- (b) $|\{a \in \partial_{-1}x, |\partial_1 a \cap U| \geq |\partial_1 a|/4\}| > k^{3/4}/2$.

It is easily seen that the process converges. Let U_∞ be the resulting subset of V , then we have

$$\text{Core}(\Phi)_{1/2} = V \setminus U_\infty. \quad (6.10)$$

We are going to show that U_∞ cannot be too large. By condition (C0), we can assume that $|U_0| \leq 2^{-0.98k}n$. Assume for contradiction that $|U_\infty| \geq 2^{-0.97kn}$ and let U be the set obtained when precisely $2^{-0.97k}n - |U_0|$ variables have been added to U_0 . By construction each variable $x \in U$ has one of the following properties.

- (00) x belongs to U_0 ,
- (0) x belongs to more than $k^{3/4}/2$ clauses a with $\partial_1 a = \{x\}$ and $|\partial_{-1} a \cap U| \geq 1$,
- (l) x belongs to more than $k^{3/4}/2$ clauses $a \in \partial_{-1,l} x$ with $|\partial_1 a \cap U| \geq l/4$.

Let $U_0 \subset U$ be the set of variables $x \in U$ that satisfy (00), $V_0 \subset U$ be the set of variables $x \in U$ that satisfy (0), and for $1 \leq l \leq k-1$ $V_l \subset U$ be the set of variables $x \in U$ that satisfy (l). As $|U| \leq |U_0| + |V_0| + \sum_{l=1}^k |V_l|$ and $|U_0| \leq |U|/k$, either $|V_0| \geq |U|/100 \geq 2^{-0.98k}n$ or there is l such that $|V_l| \geq |U|/(100l^2) \geq 2^{-0.98k}n$. Either case is impossible by a similar reasoning as previously and we obtained that $|U_\infty| \leq 2^{-0.97k}n$ w.h.p.. \square

Studying $\hat{\Phi}$ will be enough to obtain the information needed about $\tilde{\Phi}$. Indeed, we shall obtain sufficiently strong estimates of the probability of events under the random formula $\hat{\Phi}$ to transfer them into high probability statements for the biased distribution generating $\tilde{\Phi}$. More precisely, say that $\hat{\Phi}$ satisfies a property (\mathcal{P}) with very high probability (w.v.h.p.) iff (\mathcal{P}) has probability larger than $1 - \exp(-2^{-0.99k}n)$ under $\hat{\Phi}$. Then we can infer that (\mathcal{P}) has a large probability under the random formula $\tilde{\Phi}$.

Lemma 6.12. *Let \mathcal{A} be an event. Assume that $\hat{\Phi}$ satisfies \mathcal{A} w.v.h.p.. Then $\tilde{\Phi}$ satisfies \mathcal{A} w.h.p..*

Proof. Without loss of generality we can assume that \mathcal{A} contains the event

$$\{\text{all but } 2^{-0.999k}n \text{ of the } 2\ell \text{ neighborhood of variables } x \in V \text{ consists of a pure tree}\}.$$

Reformulating the definition of the planted replica model, we see that

$$\begin{aligned} \mathbb{P}[\neg \mathcal{A}] &= \frac{\sum_{\Phi} \mathbf{1}[\Phi \notin \mathcal{A}] \mathbb{P}[\hat{\Phi} = \Phi] \exp(nB_{\Phi, \ell})}{\sum_{\Phi} \mathbb{P}[\hat{\Phi} = \Phi] \exp(nB_{\Phi, \ell})} \\ &\leq \frac{\sup_{\Phi \in \mathcal{A}} \exp(nB_{\Phi, \ell})}{\sum_{\Phi} \mathbb{P}[\hat{\Phi} = \Phi] \exp(nB_{\Phi, \ell})} \mathbb{P}[\neg \mathcal{A}]. \end{aligned} \quad (6.11)$$

We observe that $B_{\Phi, \ell} \leq H(\mu_{\Phi}^{(2\ell)})$. Moreover, for all pure trees T $H(\mu_T^{(\ell)}) \leq 4^{-k}$. It therefore follows that

$$\sup_{\Phi \in \mathcal{A}} \exp(nB_{\Phi, \ell}) \leq \exp\left[nH(\mu_{\Phi}^{(2\ell)})\right] \leq \exp\left(2^{-0.999k}n\right).$$

Returning to the definition of $\mathcal{B}^{(\ell)}(k, d, \beta)$ in Section 4.2 we obtain on the other hand

$$\sum_{\Phi} \mathbb{P}[\hat{\Phi} = \Phi] \exp(nB_{\Phi, \ell}) \geq 1/2 \exp\left(n\mathcal{B}^{(\ell)}(k, d, \beta)\right) \geq \exp\left(-2^{-0.999k}n\right),$$

where the last estimate follows from an analysis similar as previously. Replacing in (6.11) yields

$$\mathbb{P}[\neg \mathcal{A}] \leq 2 \exp\left(2^{1-0.999k}n\right) \mathbb{P}[\neg \mathcal{A}] \leq 2 \exp\left(2^{1-0.999k}n - 2^{-0.99k}n\right) = o_n(1),$$

as desired. \square

In order to obtain our result, we are thus left with proving the following proposition.

Proposition 6.13. *W.v.h.p. $\hat{\Phi}$ satisfies (C0), (C1) and (C2).*

Proof of Proposition 6.4. The propositions follow from combining Lemma 6.11 combined with Lemma 6.12 and Proposition 6.13. \square

6.3. Proof of Proposition 6.13. In this section we shall study typical properties of the random formula $\widehat{\Phi}$. For a formula Φ and $0 \leq l \leq k$, we let $m_l(\Phi)$ count the number of clauses a of Φ such that $|\delta_1 a| = l$.

Lemma 6.14. *W.v.h.p. $\widehat{\Phi}$ satisfies $(\mathcal{C}0)$.*

Proof. Let Y_1 denote the number of variables $x \in V$ such that $|\{a \in \partial_{1,0} x\}| \leq 4k^{7/8}$. Let p denote the probability that a binomial of parameters $(k-1, 1-q)$ takes values 0. Using Lemma 6.1 and recalling the definition of $\tilde{p}_{k,d,\beta}^{(\ell)}$ in Section 4 gives

$$\mathbb{E}[Y_1] \leq \sum_{r=0}^{4k^{7/8}-1} \binom{d/2}{r} p^r (1-p)^{d/2-r}.$$

A simple computation reveals that $p = 2^{1-k} + \tilde{O}_k(4^{-k})$. This implies that the summand is maximal for $r = 4k^{7/8}$ and allows to bound $\mathbb{E}[Y_1]$ as

$$\mathbb{E}[Y_1] \leq O_k(k^{10}) \left(\frac{ke(\ln 2)}{k^{7/8}} \right)^{4k^{7/8}} 2^{-k} + \tilde{O}_k(4^{-k}) = \tilde{O}_k(2^{-k}).$$

A standard concentration argument then yields that $Y_1 \leq 2^{-0.99k}$ w.v.h.p..

Similarly, let Y_2 denote the number of variables $x \in V$ such that $\partial_{-1} x \geq 2$. Let Q denote the probability that a binomial of parameter $(d/2, \exp(-\beta)q^{k-1}/(1-c_\beta q^{k-1}))$ takes a value larger than 2. By another simple computation, we find $Q = \tilde{O}_k(2^{-k})$. It follows from Lemma 6.1 that $\mathbb{E}[Y_2] = Q = \tilde{O}_k(2^{-k})$. Again, by concentration this implies $Y_2 \leq 2^{-0.99k}$ w.v.h.p..

Finally, for $1 \leq l \leq k$ let $Y_3(l)$ be the number of variables $x \in V$ with $|\partial_{1,l} x| \geq k^{l+3}/l!$. By similar computations, we obtain

$$\mathbb{E}[Y_3(l)] \leq \sum_{r=k^{l+3}/l!}^{d/2} \binom{d/2}{r} \binom{k-1}{l} \frac{1}{2^k} (1 + \tilde{O}_k(2^{-k})) = \tilde{O}_k(2^{-k}).$$

It follows that $\mathbb{E}[Y_3] = \tilde{O}_k(2^{-k})$, and by the same concentration argument as previously $Y_3 \leq 2^{-0.99k}$ w.v.h.p..

The proof of the lemma is completed by noting that $|\mathbf{U}_0| \leq Y_1 + Y_2 + Y_3$. \square

We define $m'_l = \frac{1}{200} \frac{d}{2^k} k^{l+3}/l!$. The previous estimates can easily be (slightly extended and) recast as follows.

Remark 6.15. *W.v.h.p. we have for all $0 \leq l \leq k$, $m_l(\widehat{\Phi}) \leq m'_l$.*

We are now ready to complete

Lemma 6.16. *W.v.h.p. $\widehat{\Phi}$ satisfies $(\mathcal{C}1)$.*

Proof. Given $\widehat{\Phi}$, let $X_0(t, r, y)$ count the number of sets $T \subset V$ of size $|T| = tn$, such that

- $|F_0(T)| = rtn$.
- $\sum_{a \in F_0(T)} |\partial_{-1} a \cap T| = yrtkn$.

By definition of $F_1(T)$, $X_0(t, r, y) = 0$ if $y < k^{-1}$. The expected value of $X_0(t, r, y)$ can be computed in the following manner. First choose the sets T and $F_0(T)$. The latter has to be chosen among the $m_0(\widehat{\Phi})$ satisfied clauses. Among the tdn literal clones from T , choose the rtn positive literal clones that will be connected to the positive literal clones of clauses in $F_0(T)$, and the $ytdnk$ literal clones that will be connected to negative literal clones of clauses in $F_0(T)$. Make the same choices among the negative and positive literal clones of the clauses in $F_0(T)$. Then match these rtn positive literal clones (resp. $yrtkn$ negative literal clones) at random, and then match the remaining $dn/2 - rtn$ remaining positive literal clones (resp. $dn/2 - rtn$ remaining negative literal clones) at random. The normalizing factor is the total number of graphs that can be obtained from the configuration model, $(dn/2)!^2$. Without words, and using in addition Remark 6.15 to observe that we can assume $m_0(\widehat{\Phi}) \leq m'_0 = \frac{d}{2^k} k^3$, this gives

$$\begin{aligned} \mathbb{E}[X_0(t, r)] &\leq \binom{n}{tn} \binom{m'_0}{rtn} \binom{tdn}{rtn} \binom{tdn}{yrtkn} \binom{rtn}{rtn} \binom{rtn}{yrtkn} \frac{(rtnk)!(dn/2 - rtnk)!(dn/2)!}{(dn/2)!^2} \\ &\leq \binom{n}{tn} \binom{m'_0}{rtn} \binom{tdn}{rtn} \binom{tdn}{yrtkn} \binom{rtn}{rtn} \binom{rtn}{yrtkn} \left(\frac{dn/2}{rtn} \right)^{-1} \left(\frac{dn/2}{yrtkn} \right)^{-1}. \end{aligned}$$

We shall bound this quantity by using the bounds, for $1 \leq a \leq b$ and $n > 0$

$$b \ln \left(\frac{a}{b} \right) \leq \frac{1}{n} \ln \left(\frac{an}{bn} \right) \leq b \ln \left(\frac{ae}{b} \right). \quad (6.12)$$

This yields

$$\frac{1}{n} \ln \mathbb{E}[\mathbf{X}_0(t, r, y)] \leq t \ln \left(\frac{e}{t} \right) + r t \ln \left(\frac{dk^3}{2^k t r} \right) + r t \ln (2ke^2 t) + y r t k \ln \left(\frac{2e^2 t}{y} \right).$$

In particular for $r \geq k^{3/4}$, $t \in [2^{-0.98k} n, 2^{-k/20} n]$, and $y \geq 1/k$, we get

$$\frac{1}{n} \ln \mathbb{E}[\mathbf{X}_0(t, r, y)] \leq -t \ln t + t + k^{3/4} t \ln (k^{10} t) \leq -2^{-0.98k} n.$$

In particular

$$\sum_{\substack{t \in [2^{-0.98k} n, 2^{-k/20} n] \\ tn \in \mathbb{N}}} \sum_{\substack{r \in [k^{3/4}, d] \\ rtn \in \mathbb{N}}} \sum_{\substack{y \in [0, 1] \\ y rtn \in \mathbb{N}}} \mathbb{E}[\mathbf{X}_0(t, r, y)] \leq \exp \left[-2^{-0.985k} n \right].$$

This implies by Markov's inequality that w.v.h.p. there are no sets T of size $|T| \in [2^{-0.98k} n, 2^{-k/20} n]$ such that $|F_0(T)| \geq k^{3/4} |T|$. \square

Lemma 6.17. *W.v.h.p. $\hat{\Phi}$ satisfies $(\mathcal{C}2)$.*

Proof. Given $\hat{\Phi}$ and $1 \leq l \leq k$, let $\mathbf{X}_l(t, r, x, y)$ count the number of sets $T \subset V$ of size $|T| = tn$ and such that the following condition are true.

- $|F_l(T)| = rtn$.
- $\sum_{a \in F_l(T)} |\partial_1 a \cap T| = x r t k n$.
- $\sum_{a \in F_l(T)} |\partial_{-1} a \cap T| = y r t k n$.

By definition of $F_l(T)$, $\mathbf{X}_l(t, r, x, y) = 0$ if $x < lk^{-1}/4$ or $y < k^{-1}$. With Remark 6.15 we can assume

$$m_l(\hat{\Phi}) \leq m'_l.$$

Reasoning as before, we obtain

$$\mathbb{E}[\mathbf{X}_l(t, r, x, y)] \leq \binom{n}{tn} \binom{m'_l}{rtn} \binom{tdn}{x r t k n} \binom{tdn}{y r t k n} \binom{r t k n}{x r t k n} \binom{r t k n}{y r t k n} \left(\frac{dn/2}{x r t k n} \right)^{-1} \left(\frac{dn/2}{y r t k n} \right)^{-1}.$$

Taking logarithm and using (6.12), we obtain

$$\frac{1}{n} \ln \mathbb{E}[\mathbf{X}_l(t, r, x, y)] \leq t \ln \left(\frac{e}{t} \right) + r t \ln \left(\frac{dk^{l+3}}{2^k l! r t} \right) + x r t k \ln \left(\frac{2e^2 t}{x} \right) + y r t k \ln \left(\frac{2e^2 t}{y} \right).$$

In particular, for $r \geq k^{3/4}/(100l^2)$, $t \in [2^{-0.98k} n, 2^{-k/20} n]$ and $x \geq lk^{-1}/4$, $y \geq k^{-1}$, we have

$$\frac{1}{n} \ln \mathbb{E}[\mathbf{X}_l(t, r, x, y)] \leq t \ln \left(\frac{e}{t} \right) + r t \ln \left(\frac{dk^{l+6} t^{l/4+1}}{2^k r l!} \right) \leq -t \ln t + t + \frac{k^{3/4}}{100l^2} t \ln (k^{l+8} t^{l/4+1})$$

For any $1 \leq l \leq k$ we have $k^{l+6} t^{l/4+1} \leq k^{l^2}$ and we thereby obtain that

$$\frac{1}{n} \ln \mathbb{E}[\mathbf{X}_l(t, r, x, y)] \leq -2^{-0.98k} n.$$

This entails that, for any $1 \leq l \leq k$,

$$\sum_{\substack{t \in [2^{-0.98k} n, 2^{-k/20} n] \\ tn \in \mathbb{N}}} \sum_{\substack{r \in [k^{3/4}/(100l^2), d] \\ rtn \in \mathbb{N}}} \sum_{\substack{x \in [0, 1] \\ x r t k n \in \mathbb{N}}} \sum_{\substack{y \in [0, 1] \\ y r t k n \in \mathbb{N}}} \mathbb{E}[\mathbf{X}_l(t, r, x, y)] \leq \exp \left[-2^{-0.985k} n \right].$$

This implies by Markov's inequality that w.v.h.p. there are no $1 \leq l \leq k$ and no sets T of size $|T| \in [2^{-0.98k} n, 2^{-k/20} n]$ such that $|F_l(T)| \geq k^{3/4} |T|/(100l^2)$. \square

7. MOMENT COMPUTATIONS

In this section we prove Lemma 2.1, Lemma 2.2 and Proposition 4.5. We recall that $q = q(k, d, \beta)$ was defined in Section 3, Eq. (3.1).

7.1. Preliminaries. We will need the following version of the inverse function theorem.

Lemma 7.1. *Let $U \subset \mathbb{R}^h$ be an open set and let $f \in C^\infty(U)$. Assume that $u \in U$ and $r > 0$ are such that*

$$\{x \in \mathbb{R}^h : \|x - u\|_2 \leq r\} \subset U.$$

Let $Df(x)$ be the Jacobian matrix of f at x , id the identity matrix, and $\|\cdot\|$ the operator norm over $L^2(\mathbb{R}^h)$. Assume that $Df(u) = \text{id}$ and

$$\|Df(x) - \text{id}\| \leq \frac{1}{3} \quad \text{for all } x \in \mathbb{R}^h \text{ such that } \|x - u\|_2 \leq r.$$

Then for each $y \in \mathbb{R}^h$ such that $\|y - f(u)\| \leq r/2$ there is precisely one $x \in \mathbb{R}^h$ such that $\|x - u\| \leq r$ and $f(x) = y$. Furthermore, the inverse map f^{-1} is C^∞ on $\{x \in \mathbb{R}^h : \|x - u\| < r\}$, and $Df^{-1}(x) = (Df(x))^{-1}$ on this set.

We will also need the following result on the large deviation function of the multinomial distribution.

Lemma 7.2. *Let $l \geq 2$ and $(p_1, \dots, p_l) \in (0, 1)^l$ satisfying $\sum_{j=1}^l p_j = 1$ be fixed. We have, for any $(q_1, \dots, q_l) \in (0, 1)^l$ satisfying $\sum_{j=1}^l q_j = 1$*

$$\frac{1}{n} \ln P[\forall j \in [l], |\text{Multinomial}(n, p_1, \dots, p_l)_j - nq_j| \leq 0.01\sqrt{n}] = \sum_{j=1}^l q_j \ln\left(\frac{p_j}{q_j}\right) + o_n(1).$$

Finally, we will need the following concentration result.

Lemma 7.3. *Let $d \leq d_{k-\text{SAT}}$ and $\beta \in \mathbb{R}$ be fixed. For any $\alpha > 0$ there is $\delta > 0$ such that*

$$\begin{aligned} \mathbb{P}\left[\left|\frac{1}{n} \ln Z_\Phi(\beta) - \frac{1}{n} \mathbb{E} \ln [Z_\Phi(\beta)]\right| > \alpha\right] &< \exp(-\delta n), \\ \mathbb{P}\left[\left|\frac{1}{n} \ln \mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta) - \frac{1}{n} \mathbb{E} \ln [\mathcal{C}_{\hat{\Phi}, \hat{\sigma}}(\beta)]\right| > \alpha\right] &< \exp(-\delta n). \end{aligned}$$

Proof. The proof follows from the fact that if two formula Φ, Φ' differ by at most one switch of edges, the associated partition functions satisfy

$$|\ln Z_\Phi(\beta) - \ln Z_{\Phi'}(\beta)| \leq 2\beta \quad \text{and, for } \sigma \in \{-1, 1\}^n \quad |\ln \mathcal{C}_{\Phi, \sigma}(\beta) - \ln \mathcal{C}_{\Phi', \sigma}(\beta)| \leq 2\beta.$$

The stated concentration result is then a consequence of Azuma's inequality (applied to the configuration model). \square

7.2. The first moment computation. Let $z_{1,k} : \mathbb{R} \times (0, 1) \rightarrow (0, 1]$ be defined by

$$z_{1,k}(\beta, h) = 1 - c_\beta(1 - h)^k,$$

the Kullback-Leibler divergence $D_1 : (0, 1)^2 \rightarrow \mathbb{R}$ be defined by

$$D_1(\alpha, h) = \alpha \ln\left(\frac{\alpha}{h}\right) + (1 - \alpha) \ln\left(\frac{1 - \alpha}{1 - h}\right),$$

and $f_{1,k} : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f_{1,k}(d, \beta) = \ln 2 + \frac{d}{k} \ln z_{1,k}(\beta, 1 - q) + d D_1\left(\frac{1}{2}, 1 - q\right).$$

Proof of Proposition 4.5. We need to compute the expected value of $\prod_{a \in F} \psi_{a, \beta}(\sigma)$ under a random assignment $\sigma \in \{-1, 1\}^n$. To do this, we introduce a different probability space formed of all vectors in $\{-1, 1\}^{km} \times \{0, 1\}^m$

$$(\phi_{al})_{a \in [m], l \in [k]}, (y_a)_{a \in [m]}$$

with a probability distribution \mathbb{P} such that the $(\phi_{al})_{a \in [m], l \in [k]}$ are independent random variables distributed as $\mathbb{P}(\phi_{al} = 1) = 1 - q$ and the $(y_a)_{a \in [m]}$ are independent Bernoulli random variables of parameter $\exp(-\beta) / (1 + \exp(-\beta))$. We consider the two events

$$S = \{\forall a \in [m] (y_a = 0 \text{ and } \exists l \in [k], \phi_{al} = 1) \text{ or } (y_a = 1 \text{ and } \forall l \in [k], \phi_{al} = -1)\}$$

and

$$B = \left\{ \left| |\{(a, l) : \phi_{al} = 1\}| - \frac{d}{2}n \right| \leq \sqrt{n} \right\}.$$

Then, using that $\psi_{a,\beta}(\phi_{al}) = (1 + \exp(-\beta)) (\mathbb{P}[y_a = 0] \mathbf{1}_{\exists l \in [k], \phi_{al}=1} + \mathbb{P}[y_a = 0] \mathbf{1}_{\forall l \in [k], \phi_{al}=-1})$, we see the expected value of $\prod_{a \in F} \psi_{a,\beta}$ under *any* given assignment $\sigma \in \{-1, 1\}^n$ is given by $\mathbb{P}[S|B](1 + \exp(-\beta))^m$. In particular,

$$\frac{1}{n} \ln \mathbb{E}[Z_\beta(\Phi)] \sim \ln 2 + \frac{1}{n} \ln \mathbb{P}[S|B] + \frac{d}{k} \ln(1 + \exp(-\beta)). \quad (7.1)$$

By Bayes' theorem we have

$$\mathbb{P}[S|B] = \frac{\mathbb{P}[S]\mathbb{P}[B|S]}{\mathbb{P}[B]}. \quad (7.2)$$

It follows from Lemma 7.2 that

$$\frac{1}{km} \ln \mathbb{P}[B] \sim -D_1\left(\frac{1}{2}, 1 - q\right). \quad (7.3)$$

It is also straightforward to obtain that

$$\frac{1}{m} \ln \mathbb{P}[S] \sim z_{1,k}(\beta, 1 - q) - \ln(1 + \exp(-\beta)), \quad (7.4)$$

and by definition of q we have (using the central limit theorem)

$$\frac{1}{m} \ln \mathbb{P}[B|S] = o_n(1). \quad (7.5)$$

The proposition is obtained by combining Eq.(7.1-7.5). \square

7.3. The second moment computation. Recall that $c_\beta = 1 - \exp(-\beta)$. Let $\mathcal{T} = \{(h, \hat{h}) \in (0, 1)^2, \hat{h} < h\}$. Let $z_{2,k} : \mathbb{R} \times \mathcal{T} \rightarrow (0, 1]$ be defined by

$$z_{2,k}(\beta, h, \hat{h}) = 1 - 2c_\beta(1 - h)^k + c_\beta^2(1 - 2h + \hat{h})^k.$$

Lemma 7.4. *Let $g_{2,k,\beta} : \mathcal{T} \rightarrow \mathbb{R}^2$ be defined by*

$$g_{2,k,\beta}(h, \hat{h}) = \left(\frac{\hat{h} + (h - \hat{h})[1 - c_\beta(1 - h)^{k-1}]}{z_{2,k}(\beta, h, \hat{h})}, \frac{\hat{h}}{z_{2,k}(\beta, h, \hat{h})} \right).$$

Let $\alpha \in (0, 1)$ and let $\mathcal{U} = \{(x, y) \in \mathbb{R}^2, \|(x, y) - (\frac{1}{2}, \frac{1-\alpha}{2})\|_2 \leq k2^{-k}\}$. Then the equation $g_{2,k,\beta}(h, \hat{h}) = (\frac{1}{2}, \frac{1-\alpha}{2})$ admits a unique solution in $\mathcal{T} \cap \mathcal{U}$ that we denote by $(h_\beta(\alpha), \hat{h}_\beta(\alpha))$. Moreover, $\alpha \rightarrow h_\beta(\alpha)$ (resp. $\alpha \rightarrow \hat{h}_\beta(\alpha)$) is of class C^∞ on $(0, 1)$ and the following is true.

$$\hat{h}_\beta(1/2) = h_\beta(1/2)^2 = q^2, \quad (7.6)$$

$$\hat{h}'_\beta(\alpha) = h'_\beta(\alpha) - 1/2 + \tilde{O}_k(2^{-4k/3}) \quad \text{for } |\alpha - 1/2| \leq 2^{-k/3}. \quad (7.7)$$

Proof. The Jacobian matrix $Dg_{2,k,\beta}(h, \hat{h})$ of $g_{2,k,\beta}$ at $(h, \hat{h}) \in \mathcal{T}$ is given by $Dg_{2,k,\beta}(h, \hat{h}) = \text{id} + \tilde{O}_k(2^{-k})$; in particular it satisfies $\|Dg_{2,k,\beta}(h, \hat{h}) - \text{id}\| \leq 1/3$. Then Lemma 7.1 applied to $g_{2,k,\beta} \in C^\infty(\mathcal{T})$ with $y = u = (\frac{1}{2}, \frac{1-\alpha}{2})$ and $r = k2^{-k}$ imply that there is exactly one $(h_\beta(\alpha), \hat{h}_\beta(\alpha)) \in \mathcal{T}$ such that $\|(h_\beta(\alpha), \hat{h}_\beta(\alpha)) - (\frac{1}{2}, \frac{1-\alpha}{2})\|_2 \leq r$ and $g_{2,k,\beta}(h_\beta(\alpha), \hat{h}_\beta(\alpha)) = (\frac{1}{2}, \frac{1-\alpha}{2})$. Moreover, the map $\alpha \rightarrow (h_\beta(\alpha), \hat{h}_\beta(\alpha))$ is of class C^∞ and $(h'_\beta(\alpha), \hat{h}'_\beta(\alpha)) = (0, -1/2) + \tilde{O}_k(2^{-k})$. A more detailed computation (using $\frac{d}{d\alpha} g_{2,k,\beta}(h_\beta(\alpha), \hat{h}_\beta(\alpha)) = (0, -1/2)$ and the chain rule for computing derivatives) reveals that, for $|\alpha - 1/2| \leq 2^{-k/3}$

$$\begin{aligned} \hat{h}'_\beta(\alpha) + \frac{1}{2} z_{2,k}(\beta, h_\beta(\alpha), \hat{h}_\beta(\alpha)) &= \frac{1}{4} \left(h'_\beta(\alpha) \frac{\partial z_{2,k}}{\partial h} + \hat{h}'_\beta(\alpha) \frac{\partial z_{2,k}}{\partial \hat{h}} \right) (\beta, h_\beta(\alpha), \hat{h}_\beta(\alpha)) = \tilde{O}_k(4^{-k}), \\ h'_\beta(\alpha) + c_\beta \hat{h}'_\beta(\alpha) 2^{1-k} + \tilde{O}_k(2^{-4k/3}) &= \frac{1}{2} \left(h'_\beta(\alpha) \frac{\partial z_{2,k}}{\partial h} + \hat{h}'_\beta(\alpha) \frac{\partial z_{2,k}}{\partial \hat{h}} \right) (\beta, h_\beta(\alpha), \hat{h}_\beta(\alpha)) = \tilde{O}_k(4^{-k}). \end{aligned}$$

In particular

$$\hat{h}'_\beta(\alpha) - h'_\beta(\alpha) + \frac{1}{2} = \frac{1}{2} (1 - z_{2,k}(\beta, h_\beta(\alpha), \hat{h}_\beta(\alpha))) - 2^{-k} c_\beta + \tilde{O}_k(2^{-4k/3}) = \tilde{O}_k(2^{-4k/3}).$$

Finally, (7.6) is easily proved by inspection. \square

In particular, we observe that Proposition 4.5 and the above lemma imply the following.

Corollary 7.5. *We have $\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}[Z_\beta(\Phi)] = \frac{1}{2} f_{2,k}(d, \beta, 1/2)$.*

Let the Kullback-Leibler divergence $D_2 : (0, 1)^3 \rightarrow \mathbb{R}$ be defined by

$$D_2(\alpha, h, \hat{h}) = \alpha \ln \left(\frac{\alpha}{2(h - \hat{h})} \right) + \frac{1 - \alpha}{2} \ln \left(\frac{1 - \alpha}{2\hat{h}} \right) + \frac{1 - \alpha}{2} \ln \left(\frac{1 - \alpha}{2(1 - 2h + \hat{h})} \right),$$

and $f_{2,k} : \mathbb{R}^2 \times (0, 1) \rightarrow \mathbb{R}$ be defined by

$$f_{2,k}(d, \beta, \alpha) = \ln 2 + H(\alpha) + \frac{d}{k} \ln z_{2,k}(\beta, h_\beta(\alpha), \hat{h}_\beta(\alpha)) + d D_2(\alpha, h_\beta(\alpha), \hat{h}_\beta(\alpha)).$$

We also let

$$Z(d, \beta, \alpha) = \mathbb{E} \left[\sum_{\substack{\sigma, \tau \\ \sigma \cdot \tau = (2\alpha - 1)n}} \prod_{a \in F} (\psi_{a,\beta}(\sigma_a) \psi_{a,\beta}(\tau_a)) \right].$$

so that

$$\mathbb{E} [Z_\beta^2(\Phi)] = \sum_{\alpha \in \{0, 1/n, \dots, 1\}} Z(d, \beta, \alpha).$$

Proposition 7.6. *Let $d > 0, \beta \in \mathbb{R}$ and $I \subset [0, 1]$ be fixed. We have*

$$\frac{1}{n} \ln \left(\sum_{\alpha \in \{0, 1/n, \dots, 1\} \cap I} Z(d, \beta, \alpha) \right) \sim \sup_{\alpha \in I} f_{2,k}(d, \beta, \alpha)$$

and in particular

$$\frac{1}{n} \ln \mathbb{E} [Z_\beta^2(\Phi)] \sim \sup_{\alpha \in (0, 1)} f_{2,k}(d, \beta, \alpha).$$

Proof. We need to compute the expected value of $\prod_{a \in F} \psi_{a,\beta}(\sigma) \psi_{a,\beta}(\tau)$ under a random pair of assignments $(\sigma, \tau) \in \{-1, 1\}^{2n}$. To do this, we introduce a different probability space formed of all vectors in $\{-1, 1\}^{2km} \times \{0, 1\}^{2m}$

$$(\phi_{al})_{a \in [m], l \in [k]}, (y_a^{(1)}, y_a^{(2)})_{a \in [m]}$$

with a probability distribution \mathbb{P} such that the $(\phi_{al})_{a \in [m], l \in [k]}$ independent random variables satisfying

$$\phi_{al} = \begin{cases} (1, 1), & \text{with probability } \hat{h}_\beta(\alpha) \\ (1, -1), & \text{with probability } h_\beta(\alpha) - \hat{h}_\beta(\alpha) \\ (-1, 1), & \text{with probability } h_\beta(\alpha) - \hat{h}_\beta(\alpha) \\ (-1, -1), & \text{with probability } 1 - 2h_\beta(\alpha) + \hat{h}_\beta(\alpha) \end{cases}$$

independently for all a, l , and the $(y_a^{(1)})_{a \in [m]}$ (resp. $(y_a^{(2)})_{a \in [m]}$) are independent Bernoulli random variables of parameter $\exp(-\beta)/(1 + \exp(-\beta))$. We consider the following events.

$$S_1 = \{\forall a \in [m] (y_a^{(1)} = 0 \text{ and } \exists l \in [k], \phi_{al} \in \{(1, 1), (1, -1)\}) \\ \text{or } (y_a^{(1)} = 1 \text{ and } \forall l \in [k], \phi_{al} \in \{(-1, 1), (-1, -1)\})\},$$

$$S_2 = \{\forall a \in [m] (y_a^{(2)} = 0 \text{ and } \exists l \in [k], \phi_{al} \in \{(1, 1), (-1, 1)\}) \\ \text{or } (y_a^{(2)} = 1 \text{ and } \forall l \in [k], \phi_{al} \in \{(1, -1), (-1, -1)\})\},$$

$$S^{(2)} = S_1 \cap S_2,$$

and

$$B^{(2)} = \left\{ \left| |\{(a, l) : \phi_{al} = (1, 1)\}| - \frac{1 - \alpha}{2} n \right| \leq \frac{1}{\sqrt{n}}, \left| |\{(a, l) : \phi_{al} = (-1, 1)\}| - \frac{\alpha}{2} n \right| \leq \frac{1}{\sqrt{n}} \right. \\ \left. \left| |\{(a, l) : \phi_{al} = (-1, 1)\}| - \frac{\alpha}{2} n \right| \leq \frac{1}{\sqrt{n}}, \left| |\{(a, l) : \phi_{al} = (-1, -1)\}| - \frac{1 - \alpha}{2} n \right| \leq \frac{1}{\sqrt{n}} \right\}.$$

Then the expected value of $\prod_{a \in F} \psi_{a,\beta}(\sigma) \psi_{a,\beta}(\tau)$ under *any* given pair of assignments $(\sigma, \tau) \in \{-1, 1\}^{2n}$ that satisfies

$$\left| |\{i \in [n] : (\sigma_i, \tau_i) = (1, 1)\}| - \frac{1 - \alpha}{2} n \right| \leq \sqrt{n}, \left| |\{i \in [n] : (\sigma_i, \tau_i) = (1, -1)\}| - \frac{\alpha}{2} n \right| \leq \sqrt{n} \\ \left| |\{i \in [n] : (\sigma_i, \tau_i) = (-1, 1)\}| - \frac{\alpha}{2} n \right| \leq \sqrt{n}, \left| |\{i \in [n] : (\sigma_i, \tau_i) = (-1, -1)\}| - \frac{1 - \alpha}{2} n \right| \leq \sqrt{n}.$$

is given as previously by $\mathbb{P}[S^{(2)}|B^{(2)}](1 + \exp(-\beta))^{2m}$. In particular,

$$\frac{1}{n} \ln \mathbb{E} \left[Z_{\beta}^2(\Phi) \right] \sim \ln 2 + H(\alpha) + \frac{1}{n} \sup_{\alpha \in (0,1)} \ln \mathbb{P}[B^{(2)}|S^{(2)}] + \frac{2d}{k} \ln(1 + \exp(-\beta)). \quad (7.8)$$

By Bayes' theorem we have

$$\mathbb{P}[S^{(2)}|B^{(2)}] = \frac{\mathbb{P}[S^{(2)}]\mathbb{P}[B^{(2)}|S^{(2)}]}{\mathbb{P}[B^{(2)}]}. \quad (7.9)$$

It follows from Lemma 7.2 that

$$\frac{1}{km} \ln \mathbb{P}[B^{(2)}] \sim -D_2(\alpha, h_{\beta}(\alpha), \hat{h}_{\beta}(\alpha)). \quad (7.10)$$

It is also straightforward to obtain that

$$\frac{1}{m} \ln \mathbb{P}[S^{(2)}] \sim z_{2,k}(\beta, h_{\beta}(\alpha), \hat{h}_{\beta}(\alpha)) - 2 \ln(1 + \exp(-\beta)), \quad (7.11)$$

and by definition of h_{β} we have

$$\frac{1}{m} \ln \mathbb{P}[B^{(2)}|S^{(2)}] = o_n(1). \quad (7.12)$$

The proposition is obtained by combining Eq.(7.8-7.12). \square

Lemma 7.7. Assume that $d \leq d_{k-SAT}$ and $\beta \in \mathbb{R}$. Then we have

$$\sup_{\alpha \geq 2^{-k/10}} f_{2,k}(d, \beta, \alpha) \leq f_{2,k}(d, \beta, 1/2)$$

Lemma 7.8. Assume that $d \leq d_{-}(k)$ or that $d \in [d_{-}(k), d_{k-SAT}]$ and that $\beta \leq \beta_{-}(k)$. Then we have

$$\sup_{\alpha \in (0,1)} f_{2,k}(d, \beta, \alpha) \leq f_{2,k}(d, \beta, 1/2)$$

We defer the proof of these lemma to Section 7.5.

Proof of Lemma 2.1. The proposition follows by combining Corollary 7.5, Proposition 7.6 and Lemma 7.8. \square

7.4. Proof of Lemma 2.2. To facilitate the proof of Lemma 2.2 we introduce a random variable that explicitly controls the “cluster size” $\mathcal{C}_{\Phi, \sigma}(\beta)$. More precisely, we call $\sigma \in \{-1, 1\}^n$ tame in Φ iff

$$\mathcal{C}_{\Phi, \sigma}(\beta) \leq \mathbb{E}[Z_{\beta}(\Phi)].$$

Now, let

$$Z_{\text{tame}}(\Phi, \beta) = \sum_{\sigma \in \{-1, 1\}^n} \prod_{a \in F} \psi_{a, \beta}(\sigma_a) \mathbf{1}_{\sigma \text{ is tame}}.$$

We shall also need to introduce a few more notations: we denote by $\underline{m} = (m_0, \dots, m_k)$ a vector of $[m]^{k+1}$, and by $\underline{m}(\Phi) = (m_0(\Phi), \dots, m_k(\Phi))$, with the $m_j(\Phi)$ as defined in Section 6.3. Also recall that $\hat{\sigma}$ denotes the all 1 vector of length n .

Lemma 7.9. Let $d \leq d_{k-SAT}$ and $\beta \in \mathbb{R}$ be fixed. Assume that

$$\mathbb{P}[\hat{\sigma} \text{ is tame in } \hat{\Phi}] \geq \exp(o_n(n)).$$

Then

$$\mathbb{E}[Z_{\text{tame}}(\Phi, \beta)] \geq \exp(o_n(n)) \mathbb{E}[Z_{\beta}(\Phi)].$$

Proof. Given that $\underline{m}(\Phi) = \underline{m}(\hat{\Phi})$ the two formula Φ and $\hat{\Phi}$ are identically distributed. Thus we have for any $\underline{m} \in [m]^{k+1}$

$$\mathbb{P}[\hat{\sigma} \text{ is not a tame in } \Phi | \underline{m}(\Phi) = \underline{m}] = \mathbb{P}[\hat{\sigma} \text{ is not a tame in } \hat{\Phi} | \underline{m}(\hat{\Phi}) = \underline{m}].$$

In particular this implies that

$$\begin{aligned} \mathbb{E}[Z_{\beta}(\Phi) - Z_{\text{tame}}(\Phi, \beta)] &= 2^n \sum_{\underline{m} \in [m]^{k+1}} \mathbb{P}[\hat{\sigma} \text{ is not a tame in } \Phi | \underline{m}(\Phi) = \underline{m}] \mathbb{P}[\underline{m}(\Phi) = \underline{m}] \exp(-\beta m_0) \\ &\leq 2^n \sum_{\underline{m} \in [m]^{k+1}} \mathbb{P}[\hat{\sigma} \text{ is not a tame in } \hat{\Phi} | \underline{m}(\hat{\Phi}) = \underline{m}] \mathbb{P}[\underline{m}(\hat{\Phi}) = \underline{m}] \mathbb{E}[Z_{\beta}(\Phi)] \\ &\leq \mathbb{P}[\hat{\sigma} \text{ is not a tame in } \hat{\Phi}] \mathbb{E}[Z_{\beta}(\Phi)]. \end{aligned}$$

This concludes the proof of the lemma. □

Lemma 7.10. Assume that $d \leq d_{k-SAT}$ and $\beta \in \mathbb{R}$ are such that

$$\frac{\mathbb{E}[Z_{\text{tame}}(\Phi, \beta)]}{\mathbb{E}[Z_\beta(\Phi)]} > \exp(o_n(n)).$$

Then

$$\frac{\mathbb{E}[Z_{\text{tame}}(\Phi, \beta)]^2}{\mathbb{E}[Z_{\text{tame}}(\Phi, \beta)^2]} > \exp(o_n(n)).$$

Proof. We let

$$Z_{\text{tame}}(d, \beta, \alpha) = \mathbb{E} \left[\sum_{\substack{\sigma, \tau \\ \sigma \cdot \tau = (2\alpha - 1)n}} \prod_{a \in F} (\psi_{a, \beta}(\sigma_a) \psi_{a, \beta}(\tau_a)) \mathbf{1}_{\sigma \text{ is a tame}} \mathbf{1}_{\tau \text{ is a tame}} \right].$$

Then we have, by the definition of a “tame”

$$\begin{aligned} \sum_{\alpha \leq 2^{-k/10}} Z_{\text{tame}}(d, \beta, \alpha) &\leq \mathbb{E} \left[\sum_{\sigma \in \{-1, 1\}^n} \prod_{a \in F} \psi_{a, \beta}(\sigma_a) \mathbf{1}_{\sigma \text{ is a tame}} \mathcal{C}_{\Phi, \sigma}(\beta) \right] \\ &\leq \mathbb{E} [Z_\beta(\Phi)]^2 \end{aligned}$$

On the other hand we have with Lemma 7.7

$$\sum_{\alpha \geq 2^{-k/10}} Z_{\text{tame}}(d, \beta, \alpha) \leq \sum_{\alpha \geq 2^{-k/10}} Z(d, \beta, \alpha) = \exp(o_n(n)) \mathbb{E} [Z_\beta(\Phi)]^2.$$

This implies that

$$\begin{aligned} \mathbb{E}[Z_{\text{tame}}(\Phi, \beta)^2] &= \sum_{\alpha < 2^{-k/10}} Z_{\text{tame}}(d, \beta, \alpha) + \sum_{\alpha \geq 2^{-k/10}} Z_{\text{tame}}(d, \beta, \alpha) \\ &= \exp(o_n(n)) O \left(\mathbb{E} [Z_\beta(\Phi)]^2 \right). \end{aligned}$$

The lemma then follows from the assumption that $\mathbb{E} [Z_\beta(\Phi)] \leq \exp(o_n(n)) (\mathbb{E} [Z_{\text{tame}}(\Phi, \beta)])$. □

The reverse direction of Lemma 2.2 will be given by the following lemma.

Lemma 7.11. Assume that $d \in [d_-(k), d_{k-SAT}]$ and $\beta > \beta_-(k)$ are such that (2.3) holds. Then

$$\frac{1}{n} \mathbb{E} \ln [Z_\beta(\Phi)] \sim \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)].$$

Proof. We can apply Lemma 7.9 to find that

$$\frac{\mathbb{E}[Z_{\text{tame}}(\Phi, \beta)]}{\mathbb{E}[Z_\beta(\Phi)]} \geq \exp(o_n(n)).$$

Hence Lemma 7.10 implies that

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}[Z_{\text{tame}}(\Phi, \beta)]^2}{\mathbb{E}[Z_{\text{tame}}(\Phi, \beta)^2]} > \exp(o_n(n)).$$

Using the Paley-Zigmond inequality we have

$$\liminf_{n \rightarrow \infty} \mathbb{P} [Z_{\text{tame}}(\Phi, \beta) \geq \mathbb{E} [Z_{\text{tame}}(\Phi, \beta)] / 2] \geq \exp(o_n(n)).$$

In particular, we have

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left[Z_\beta(\Phi) \geq \frac{c_1}{2} \mathbb{E} [Z_\beta(\Phi)] \right] \geq \exp(o_n(n)).$$

In other words

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \ln Z_\beta(\Phi) \geq \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)] - o_n(1) \right] \geq \exp(o_n(n)).$$

It follows from Lemma 7.3 that

$$\frac{1}{n} \mathbb{E} \ln [Z_\beta(\Phi)] \geq \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)] - o_n(1).$$

The proof is completed by Jensen's inequality which give us

$$\frac{1}{n} \mathbb{E} \ln [Z_\beta(\Phi)] \leq \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)].$$

□

The second part of the proposition will be a simple application of the following lemma, which is similar to Lemma 7.9.

Lemma 7.12. *Let $d \leq d_{k-SAT}$ and $\beta \in \mathbb{R}$ be fixed. Assume that there exists a sequence of event $\widehat{\mathcal{E}}_n$ such that*

$$\mathbb{P} [\Phi \in \widehat{\mathcal{E}}_n] = 1 - o_n(1) \quad \text{and} \quad \liminf_{n \rightarrow \infty} \mathbb{P} [\widehat{\Phi} \in \widehat{\mathcal{E}}_n]^{1/n} < 1.$$

Then we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)] < \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)].$$

Proof. Let $m(n)$ be a monotonically increasing sequence and ξ be such that $\mathbb{P} [\widehat{\Phi} \in \widehat{\mathcal{E}}_n]^{1/m(n)} \leq \exp(-\xi m(n))$. We have, by the same steps as in the proof of Lemma 7.9

$$\mathbb{E} [Z_\beta(\Phi) \mathbf{1}_{\Phi \in \widehat{\mathcal{E}}_{m(n)}}] \leq \mathbb{P} [\widehat{\Phi} \in \widehat{\mathcal{E}}_{m(n)}] \mathbb{E} [Z_\beta(\Phi)] \leq \exp(-\xi m(n)) \mathbb{E} [Z(\Phi(m(n), k, d), \beta)].$$

Thereby we obtain, using that $\frac{1}{n} \ln Z_\beta(\Phi) \in [-\beta + \ln 2, \ln 2]$ for all Φ

$$\begin{aligned} \frac{1}{m(n)} \mathbb{E} \ln [Z_\beta(\Phi)] &= \frac{1}{m(n)} \mathbb{E} \ln [Z_\beta(\Phi) \mathbf{1}_{\Phi \in \widehat{\mathcal{E}}_{m(n)}}] + o_n(1) \\ &\leq \frac{1}{m(n)} \ln \mathbb{E} [Z_\beta(\Phi) \mathbf{1}_{\Phi \in \widehat{\mathcal{E}}_{m(n)}}] + o_n(1) \\ &\leq \frac{1}{m(n)} \mathbb{E} \ln [Z_\beta(\Phi)] - \xi + o_n(1). \end{aligned}$$

□

Proof of Lemma 2.2. Assume that Eq. (2.3) holds. Then by Lemma 7.11 we have $\frac{1}{n} \mathbb{E} \ln [Z_\beta(\Phi)] \sim \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)]$.

Assume that Eq. (2.3) does not hold and let ϵ be such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} [\ln \mathcal{C}_{\widehat{\Phi}, \widehat{\sigma}}(\beta)] \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)] + \epsilon$$

Let $z = \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)] + \epsilon/2$ and \mathcal{E}_n be the event that $\frac{1}{n} \ln Z_\beta(\Phi) > \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)] + \epsilon/2$. Then using Jensen's inequality and Lemma 7.3 we obtain $\mathbb{P} [\Phi \in \mathcal{E}_n]^{1/n} \sim 1$ while $\liminf_{n \rightarrow \infty} \mathbb{P} [\widehat{\Phi} \in \mathcal{E}_n]^{1/n} < 1$. Therefore with Lemma 7.12 we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \ln [Z_\beta(\Phi)] < \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E} [Z_\beta(\Phi)].$$

□

7.5. Proof of Lemma 7.7 and Lemma 7.8. We first need to study $f_{2,k}(d, \beta, \cdot)$ locally around $\alpha = 1/2$ and compare it with $f_{1,k}(d, \beta)$. This will be given by the two following lemmas.

Lemma 7.13. *We have, for $d \leq d_{k-SAT}$ and $\beta \in \mathbb{R}$,*

$$f_{1,k}(d, \beta) = \ln 2 - \frac{d}{k} \left(c_\beta 2^{-k} + 2^{-1-2k} - k 2^{-1-2k} \right) + \tilde{O}_k(4^{-k}).$$

Proof. The result follows from a direct computation, using the observation that $q = \frac{1}{2} + c_\beta 2^{-1-k} + \tilde{O}_k(4^{-k})$. □

Lemma 7.14. *Let $d \leq d_{k-SAT}$ and $\beta \in \mathbb{R}$ be fixed. $f_{2,k}$ is of class C^∞ on $\mathbb{R}^2 \times (0, 1)$. It satisfies $f_{2,k}(d, \beta, 1/2) = 2f_{1,k}(d, \beta)$, $\frac{\partial}{\partial \alpha} f_{2,k}(d, \beta, 1/2) = 0$ and $\sup_{|\alpha - 1/2| \leq 2^{-k/3}} \frac{\partial^2}{\partial \alpha^2} f_{2,k}(d, \beta, \alpha) < 0$.*

Proof. $\alpha \rightarrow H(\alpha)$ is clearly of class C^∞ on $(0, 1)$. Similarly, $(\beta, h, \hat{h}) \rightarrow \ln z_{2,k}(\beta, h, \hat{h})$ is of class C^∞ on \mathcal{T} and D is of class C^∞ on $(0, 1) \times \mathcal{T}$. The smoothness of $\alpha \rightarrow f_{2,k}(d, \beta, \alpha)$ therefore follows from the one of $\alpha \mapsto (h_\beta(\alpha), \hat{h}_\beta(\alpha))$ granted by Lemma 7.4.

Because $(h_\beta(\alpha), \hat{h}_\beta(\alpha))$ satisfy $g_{2,k,\beta}(h_\beta(\alpha), \hat{h}_\beta(\alpha)) = (\frac{1}{2}, \frac{1-\alpha}{2})$, we have using the chain rule

$$\begin{aligned}\frac{\partial z_{2,k}}{\partial h}(\beta, h_\beta(\alpha), \hat{h}_\beta(\alpha)) &= \frac{\partial D_2}{\partial h}(\alpha, h_\beta(\alpha), \hat{h}_\beta(\alpha)), \\ \frac{\partial z_{2,k}}{\partial \hat{h}}(\beta, h_\beta(\alpha), \hat{h}_\beta(\alpha)) &= \frac{\partial D_2}{\partial \hat{h}}(\alpha, h_\beta(\alpha), \hat{h}_\beta(\alpha)).\end{aligned}$$

The differential of $f_{2,k}$ with respect to α then simplifies to

$$\begin{aligned}\frac{\partial f_{2,k}}{\partial \alpha}(d, \beta, \alpha) &= H'(\alpha) + d \frac{\partial}{\partial \alpha} D_2(\alpha, h_\beta(\alpha), \hat{h}_\beta(\alpha)) \\ &= H'(\alpha) - d H'(\alpha) + \frac{d}{2} \ln \left(\frac{\hat{h}_\beta(\alpha)(1 - 2h_\beta(\alpha) + \hat{h}_\beta(\alpha))}{(h_\beta(\alpha) - \hat{h}_\beta(\alpha))^2} \right).\end{aligned}$$

In particular, for $\alpha = 1/2$ we have $\hat{h}_\beta(1/2) = h_\beta(1/2)^2$ and $\frac{\partial f_{2,k}}{\partial \alpha}(d, \beta, 1/2) = 0$.

Differentiating once more with respect to α yields

$$\begin{aligned}\frac{\partial^2 f_{2,k}}{\partial^2 \alpha}(d, \beta, \alpha) &= H''(\alpha) + d \frac{\partial^2}{\partial \alpha^2} D_2(\alpha, h_\beta(\alpha), \hat{h}_\beta(\alpha)) + d \frac{\partial^2}{\partial \alpha \partial h} D_2(\alpha, h_\beta(\alpha), \hat{h}_\beta(\alpha)) + d \frac{\partial^2}{\partial \alpha \partial \hat{h}} D_2(\alpha, h_\beta(\alpha), \hat{h}_\beta(\alpha)) \\ &= H''(\alpha) - d H''(\alpha) + \frac{d}{2} \left[-\frac{2}{h_\beta(\alpha) - \hat{h}_\beta(\alpha)} - \frac{2}{1 - 2h_\beta(\alpha) + \hat{h}_\beta(\alpha)} \right] h'_\beta(\alpha) \\ &\quad + \frac{d}{2} \left[\frac{1}{\hat{h}_\beta(\alpha)} + \frac{1}{1 - 2h_\beta(\alpha) + \hat{h}_\beta(\alpha)} + \frac{2}{h_\beta(\alpha) - \hat{h}_\beta(\alpha)} \right] \hat{h}'_\beta(\alpha).\end{aligned}$$

In particular for $|\alpha - 1/2| \leq 2^{-k/3}$ we have with Lemma 7.4

$$\begin{aligned}\frac{\partial^2 f_{2,k}}{\partial^2 \alpha}(d, \beta, 1/2) &= H''(\alpha) - d H''(\alpha) - 8d h'_\beta(\alpha) + 8d \hat{h}'_\beta(\alpha) + \tilde{O}_k(2^{-k/3}) \\ &= -4 + 8d \left[\hat{h}'_\beta(\alpha) - h'_\beta(\alpha) + \frac{1}{2} \right] + \tilde{O}_k(2^{-k/3}) \\ &= -4 + \tilde{O}_k(2^{-k/3}).\end{aligned}$$

□

We now study $f_{2,k}(d, \beta, \alpha)$ when $|\alpha - 1/2| > k^2 2^{-k/2}$. For the sake of readability, we decompose this study in small steps. We first show that we can upper bound $f_{2,k}$ by a simpler function. Let $\bar{z}_{2,k} : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$ and $\bar{f}_{2,k} : \mathbb{R}^2 \times (0, 1) \rightarrow \mathbb{R}$ be defined by

$$\bar{z}_{2,k}(\beta, \alpha) = 1 - 2c_\beta 2^{-k} + c_\beta^2 \left(\frac{1-\alpha}{2} \right)^k, \quad \bar{f}_{2,k}(d, \beta, \alpha) = \ln 2 + H(\alpha) + \frac{d}{k} \ln(\bar{z}_{2,k}(d, \beta, \alpha)).$$

Lemma 7.15. *For all $\alpha \in (0, 1)$ we have $f_{2,k}(d, \beta, \alpha) \leq \bar{f}_{2,k}(d, \beta, \alpha)$.*

Proof. Consider the event $B^{(2)}, S^{(2)}$ defined in the proof of Proposition 7.6 and their probability \mathbb{P}' under the distribution where (with the notations of the proof of Proposition 7.6)

$$\phi_{al} = \begin{cases} (1, 1), & \text{with probability } (1 - \alpha)/2 \\ (1, -1), & \text{with probability } \alpha/2 \\ (-1, 1), & \text{with probability } \alpha/2 \\ (-1, -1), & \text{with probability } (1 - \alpha)/2 \end{cases}$$

independently for all $a \in [m], l \in [k]$. We have

$$\frac{1}{m} \ln \mathbb{P}'[S^{(2)}] \sim \ln \left(1 - 2c_\beta 2^{-k} + c_\beta^2 \left(\frac{1-\alpha}{2} \right) \right) - \ln(1 + \exp(-\beta)),$$

$$\begin{aligned}\frac{1}{m} \ln \mathbb{P}'[B^{(2)}] &\sim 1, \\ \frac{1}{m} \ln \mathbb{P}'[S^{(2)}|B^{(2)}] &\sim \frac{1}{m} \ln \mathbb{P}[S^{(2)}|B^{(2)}].\end{aligned}$$

In particular, with Bayes' theorem

$$\bar{f}_{2,k}(d, \beta, \alpha) - f_{2,k}(d, \beta, \alpha) \sim \frac{1}{m} \ln \mathbb{P}'[S^{(2)}] - \frac{1}{m} \ln \mathbb{P}[S^{(2)}|B^{(2)}] \sim -\frac{1}{m} \mathbb{P}'\beta[B^{(2)}|S^{(2)}] \geq 0$$

□

Lemma 7.16. Assume that $d \leq d_{k-\text{SAT}}$ and $\beta \in \mathbb{R}$. Then

$$\bar{f}_{2,k}(d, \beta, 1/2 - k^2 2^{-k/2}) < f_{2,k}(d, \beta, 1/2).$$

Proof. We compute

$$\bar{f}_{2,k}(d, \beta, 1/2 - k^2 2^{-k/2}) = 2 \ln 2 - k^4 2^{1-k} + \frac{d}{k} \left(-c_\beta 2^{1-k} - c_\beta^2 2^{-2k} \right) + \bar{O}_k(2^{-4k/3})$$

On the other hand using Lemma 7.14 and Lemma 7.13 we have

$$f_{2,k}(d, \beta, 1/2) = 2f_{1,k}(d, \beta) = 2 \ln 2 - 2 \frac{d}{k} \left(c_\beta 2^{-k} + 2^{-1-2k} - k 2^{-1-2k} \right) + \bar{O}_k(4^{-k}).$$

It follows that

$$\bar{f}_{2,k}(d, \beta, 1/2 - k^2 2^{-k/2}) - f_{2,k}(d, \beta, 1/2) \leq -k^4 2^{1-k} + d O_k(4^{-k}) + \bar{O}_k(4^{-k}) < 0.$$

□

In order to prove Lemma 7.7-7.8, it will be convenient to restrict the range of (d, β) that we need to consider. We define $\bar{f}_{2,k} : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$ and $\bar{z}_{2,k} : (0, 1) \rightarrow \mathbb{R}$ by

$$\bar{f}_{2,k}(d, \alpha) = \lim_{\beta \rightarrow \infty} \bar{f}_{2,k}(d, \beta, \alpha), \quad \bar{z}_{2,k}(\alpha) = \lim_{\beta \rightarrow \infty} \bar{z}_{2,k}(\beta, \alpha).$$

The following claim is immediate, once one observes that $\frac{\partial}{\partial d} \bar{f}_{2,k}(d, \alpha) = \frac{1}{k} \ln(\bar{z}_{2,k}(d, \beta, \alpha))$.

Claim 7.17. Assume that $d \leq d_{k-\text{SAT}}$. Then for $\alpha \in (0, 1/2 - k^2 2^{-k/2})$ we have

$$\frac{\partial}{\partial d} \bar{f}_{2,k}(d, \alpha) \geq \frac{\partial}{\partial d} \bar{f}_{2,k}(d, 1/2 - k^2 2^{-k/2}).$$

Similarly, we have the following.

Claim 7.18. Assume that $d \leq d_{k-\text{SAT}}$. Then for $\alpha \in (0, 1/2 - k^2 2^{-k/2})$ we have

$$\frac{\partial}{\partial \beta} \bar{f}_{2,k}(d, \beta, \alpha) \geq \frac{\partial}{\partial \beta} \bar{f}_{2,k}(d, \beta, 1/2 - k^2 2^{-k/2}).$$

Proof. We compute

$$\frac{\partial}{\partial \beta} \bar{f}_{2,k}(d, \beta, \alpha) = -\frac{\exp(-\beta)}{2^{k-1}} \frac{d}{k} \frac{1 - c_\beta (1 - \alpha)^k}{1 - 2c_\beta 2^{-k} + c_\beta^2 \left(\frac{1-\alpha}{2}\right)^k}.$$

In particular

$$\frac{\partial^2}{\partial \alpha \partial \beta} \bar{f}_{2,k}(d, \beta, \alpha) = -\frac{d \exp(-\beta)}{2^{k-1}} c_\beta (1 - \alpha)^{k-1} (1 + \bar{O}_k(2^{-k})) < 0.$$

□

Therefore, in order to prove Lemma 7.7 we can assume that $d = d_{k-\text{SAT}}$ and $\beta \rightarrow \infty$, and to prove Lemma 7.8, we can focus on the following two cases.

- $d = d_-(k)$ and $\beta \rightarrow \infty$,
- $d = d_{k-\text{SAT}}$ and $\beta = \beta_-(k)$.

Lemma 7.19. *We have*

$$\sup_{\alpha \in [2^{-k+10}, 1/2 - k^2 2^{-k/2}]} \bar{f}_{2,k}(d_{k-\text{SAT}}, \alpha) \leq \bar{f}_{2,k}(d_{k-\text{SAT}}, 1/2 - k^2 2^{-k/2}).$$

Proof. We first compute

$$\bar{f}_{2,k}(d_{k-\text{SAT}}, 1/2 - k^2 2^{-k/2}) = -k^4 2^{1-k} + \tilde{O}_k(4^{-k}). \quad (7.13)$$

We differentiate $\bar{f}_{2,k}(d_{k-\text{SAT}}, \alpha)$ with respect to α .

$$\frac{\partial \bar{f}_{2,k}}{\partial \alpha}(d_{k-\text{SAT}}, \alpha) = -\ln\left(\frac{\alpha}{1-\alpha}\right) - \frac{d_{k-\text{SAT}}}{2^k} \frac{(1-\alpha)^{k-1}}{\bar{z}_{2,k}(\alpha)}.$$

Assume that $\alpha \in [1/2 - k^2 2^{k/2}, 1/2 - 2^{-k/3}]$. Then we have

$$\frac{\partial \bar{f}_{2,k}}{\partial \alpha}(d_{k-\text{SAT}}, \alpha) \geq -\ln\left(\frac{1/2 - k^2 2^{-k/2}}{1/2 + k^2 2^{-k/2}}\right) - (k \ln 2) 2^{1-k} (1 + O_k(2^{-k})) > 0. \quad (7.14)$$

Assume that $\alpha \in [0.4, 1/2 - 2^{-k/3}]$. Then we have

$$\frac{\partial \bar{f}_{2,k}}{\partial \alpha}(d_{k-\text{SAT}}, \alpha) \geq -\ln\left(\frac{1/2 - 2^{-k/3}}{1/2 + 2^{-k/3}}\right) - (k \ln 2) (0.6)^{k-1} (1 + O_k(2^{-k})) > 0. \quad (7.15)$$

Similarly, for $\alpha \in [2(\ln k)/k, 0.4]$, we have

$$\frac{\partial \bar{f}_{2,k}}{\partial \alpha}(d_{k-\text{SAT}}, \alpha) \geq -\ln\left(\frac{0.4}{0.6}\right) - \frac{\ln 2}{k} + O_k((\ln k) k^{-2}) > 0. \quad (7.16)$$

For $\alpha \in [2^{-k/10}, 2(\ln k)/k]$, we compute with the help of (7.13), and using $-(1 - \exp(-x)) \leq -x/2$ for $0 < x < 1$

$$\bar{f}_{2,k}(d_{k-\text{SAT}}, \alpha) \leq \alpha \left(-\ln \alpha - \frac{k \ln 2}{2}\right) + O_k(\alpha) < \bar{f}_{2,k}(d_{k-\text{SAT}}, 1/2 - k^2 2^{-k/2}). \quad (7.17)$$

The lemma follows from Eq.(7.14-7.16) and (7.17). □

Lemma 7.20. *We have*

$$\sup_{\alpha \in (0, 1/2 - k^2 2^{-k/2}]} \bar{f}_{2,k}(d_-(k), \alpha) \leq \bar{f}_{2,k}(d_-(k), 1/2 - k^2 2^{-k/2}).$$

Proof. We first compute

$$\bar{f}_{2,k}(d_-(k), 1/2 - k^2 2^{-k/2}) = k^5 2^{1-k} + O_k(k^4 2^{-k}). \quad (7.18)$$

By Lemma 7.19 and Claim 7.17 we also have

$$\sup_{\alpha \in (2^{-k/10}, 1/2 - k^2 2^{-k/2}]} \bar{f}_{2,k}(d_-(k), \alpha) \leq \bar{f}_{2,k}(d_-(k), 1/2 - k^2 2^{-k/2}). \quad (7.19)$$

Let α^* be a maximum of $\bar{f}_{2,k}(d_-(k), \cdot)$ over $(0, 2^{-k/10})$. The equation $\frac{\partial \bar{f}_{2,k}(d_-(k), \alpha)}{\partial \alpha} = 0$ reads

$$-\ln\left(\frac{\alpha^*}{1-\alpha^*}\right) = k \ln 2 (1 + o_k(1)). \quad (7.20)$$

Expanding for $\alpha \leq 2^{-k/10}$, we obtain that $\alpha^* \sim 2^{-k}$. Using (7.20) once again yields

$$\alpha^* = 2^{-1-k} + \tilde{O}_k(4^{-k}).$$

In particular

$$\begin{aligned} \bar{f}_{2,k}(d_-(k), \alpha^*) &= \ln 2 + k 2^{-k} + 2^{-k} + \frac{d_-(k)}{k} \left(-2^{-k} - 2^{-1-2k} - k 2^{-2k}\right) + \tilde{O}_k(4^{-k}) \\ &= k^5 2^{-k} + \tilde{O}_k(4^{-k}). \end{aligned}$$

Noting that $\lim_{\alpha \rightarrow 0} \frac{\partial}{\partial \alpha} \bar{f}_{2,k}(d_-(k), \alpha) = \infty$ and using (7.18), this gives

$$\sup_{\alpha \in (0, 2^{-k/10})} \bar{f}_{2,k}(d_-(k), \alpha) \leq \bar{f}_{2,k}(d_-(k), 1/2 - k^2 2^{-k/2}). \quad (7.21)$$

Collecting (7.19) and (7.21) ends the proof of the lemma. □

Lemma 7.21. *We have*

$$\sup_{\alpha \in (0, 1/2 - k^2 2^{-k/2}]} \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), \alpha) \leq \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), 1/2 - k^2 2^{-k/2}).$$

Proof. By combining Lemma 7.19, Claim 7.17 and Claim 7.18 we obtain

$$\sup_{\alpha \in [2^{-k/10}, 1/2 - k^2 2^{-k/2}]} \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), \alpha) = \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), 1/2 - k^2 2^{-k/2}). \quad (7.22)$$

Let α^* be a maximum of $\tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), \cdot)$ over $(0, 2^{-k/10})$. The equation $\frac{\partial \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), \alpha)}{\partial \alpha} = 0$ is again given by (7.20) and hence

$$\alpha^* = \frac{1}{2} - 2^{-1-k} + \tilde{O}_k(4^{-k}).$$

In particular

$$\begin{aligned} \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), \alpha^*) &= \ln 2 + k2^{-k} + 2^{-k} + \frac{d_{k-\text{SAT}}}{k} \left(-c_\beta 2^{-k} - 2^{-1-2k} - k2^{-2k} \right) + \tilde{O}_k(4^{-k}) \\ &= \tilde{O}_k(4^{-k}), \end{aligned}$$

while

$$\tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), 1/2 - k^2 2^{-k/10}) = k^{10} 2^{1-k} + O_k(k^4 2^{-k}).$$

Noting that $\lim_{\alpha \rightarrow 0} \frac{\partial}{\partial \alpha} \tilde{f}_{2,k}(d, \beta_-(k), \alpha) = \infty$, this gives

$$\sup_{\alpha \in (0, 2^{-k/10})} \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), \alpha) < \tilde{f}_{2,k}(d_{k-\text{SAT}}, \beta_-(k), 1/2 - k^2 2^{-k/2}). \quad (7.23)$$

This concludes the proof of the lemma. \square

Proof of Lemma 7.7. Let d and β be as in Lemma 7.7. We first observe that for $\alpha \in (0, 1/2)$, $f_{2,k}(d, \beta, \alpha) \geq f_{2,k}(d, \beta, 1 - \alpha)$. Therefore, we can restrict ourselves to $\alpha \in (0, 1/2)$. By Lemma 7.14 we have

$$\sup_{1/2 - 2^{-k/3} \leq \alpha \leq 1/2} f_{2,k}(d, \beta, \alpha) \leq f_{2,k}(d, \beta, 1/2).$$

Combining Claim 7.17-7.17 with Lemma 7.19 we obtain

$$\sup_{\alpha \in [2^{-k/10}, 1/2 - k^2 2^{-k/2}]} \tilde{f}_{2,k}(d, \beta, \alpha) \leq \tilde{f}_{2,k}(d, \beta, 1/2 - k^2 2^{-k/2}).$$

Using in addition Lemma 7.15 and Lemma 7.16 ends the proof of the lemma. \square

Proof of Lemma 7.8. The proof follows by combining the previous results with Lemma 7.20 and Lemma 7.21. \square

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